Study Guide for Final Exam

1. You are supposed to be able to determine the domain of a function, looking at the conditions for its expression to be well-defined. Some examples of the conditions are:

- What is inside of the square root must be non-negative (where there is no restriction on what is inside of the cubic root).
- The denominator must not be zero.
- What is inside of the logarithm must be strictly positive.

Example Problems:

1.1. Find the domain of the following function.

(i) $f(t) = \sqrt{5-t} + \dfrac{1}{\sqrt{t^2-4}}$.

(ii) $f(x) = \dfrac{1}{\ln(x^2-1)}$.

(iii) $f(x) = \sqrt{e^{2x} - 2e^x + \dfrac{3}{4}}$.

(iv) $f(x) = \sqrt{\dfrac{1}{2 - \ln(x-1)}}$.

2. Having the information on the range of a given rotation angle θ and knowing the value of a trigonometric function, you are supposed to be able to determine the values of the other trigonometric functions.

Example Problems:

2.1. We have the information

$$\sin\theta = -\frac{12}{13} \quad \text{and} \quad \pi < \theta < \frac{3\pi}{2}.$$

Determine the value of $\cot\theta$.

2.2. We have the information

$$\sin\theta = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{3\pi}{2} < \theta < 2\pi.$$

Determine the value of $\tan\theta$.

2.3. We have the information

$$\cos 2\theta = -\frac{1}{2} \quad \text{and} \quad 0 < \theta < \pi.$$

Determine the value of $\sin\theta$.

3. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formulas for sine and cosine, $\sin^2 x + \cos^2 x = 1$, etc.).

Example Problems:

3.1. How many solutions are there on the interval $[0, 2\pi]$ for the equation

$$\sqrt{3}\sin x = \sin(2x) \quad ?$$

3.2. Find all the values x on the interval $[0, 2\pi]$ satisfying the equation

$$\cos(2x) - \sin(2x) = 0.$$

4. Given a function which is one-to-one, with a given domain and range, you are supposed to be able to find the formula for its inverse function, describing its domain and range. You should also understand th relation between the graph of the original function and the one for its inverse.

Example Problems:

4.1. Find a formula for the inverse of the following function, and describe the domain and range of the inverse function.

(i) $f(x) = \dfrac{6x-1}{2x+1}$.

(ii) $f(x) = \dfrac{2e^x-1}{2e^x+1}$.

(iii) $f(x) = 1 - \sqrt{x+1}$.

5. When a function is defined piecewise and depending on some variables, you are supposed to know how to determine those variables so that the function becomes continuous entirely over its domain.

Example:

5.1. Look at Webassign HW 5 Problem 18, HW 6 Problem 1.

5.2. Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 - a & \text{if } x \leq 1 \\ \dfrac{3x^2+12x-b}{x^2+2x-3} & \text{if } x > 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

6. Starting with the graph of a given function, you are supposed to be able to derive the equation of the graph resulting from horizontal/vertical shifts, reflections with respect to the lines.

Example Problems:

6.1. Starting with the graph of a function $y = e^{3x+1}$, write down the equation of the graph that results from
(i) reflecting with respect to the line $x = -2$,
(ii) reflecting with respect to the line $y = 5$,
(ii) reflecting with respect to the line $y = x$.

6.2. Starting with the graph of $y = e^{2x-1}$, we reflect it with respect to $x = a$. The equation of the resulting graph is $y = e^{-2x+9}$. Determine the value of a.

7. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

Example Problems:

7.1. Suppose we have a function $f(x)$ with $f'(2) = 5$.
Determine the following values:
(i) $g'(1)$ where $g(x) = f(2x)$.
(ii) $\lim_{h\to 0} \dfrac{f(2+4h) - f(2)}{3h}$.
(iii) $\lim_{h\to 0} \dfrac{f(2+4h) - f(2+5h)}{9h}$.

8. You are supposed to be able to compute the derivative of a function, and understand that its value represents the slope of the tangent line to the graph of the function.

Example Problems:

8.1. Find the equation of the line that is tangent to the curve $y = \dfrac{2}{3}x\sqrt{x}$ and is also parallel to the line $y = 2x + 3$.
8.2. Find the equation(s) of the tangent line(s) to the graph of a function $y = x^2$, passing through the point $(1, -3)$.

9. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions.

Example problems:

9.1. Compute the derivative of the following function:

(i) $y = \sin(\sin(\sin x))$
(ii) $y = \left(\dfrac{t-2}{2t+1}\right)^9$
(iii) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
(iv) $y = e^{\sec 3\theta}$
(v) $y = e^{2^{x^3}}$

9.2. Suppose that $F(x) = f(x)^2 \cdot f(g(x))$ and that the functions f and g satisfy the following conditions:

$$\begin{cases} f(1) = 5, & f(2) = 3, & f(3) = -1 \\ f'(1) = 4, & f'(2) = 3, & f'(3) = -2 \\ g(1) = 3, & g(2) = 2, & g(3) = -1 \\ g'(1) = 2, & g'(2) = 3, & g'(3) = 4 \end{cases}$$

Find $F'(1)$.

9.3. We have two everywhere differentiable functions h and g such that $x^3 = g(h(x))$ and that $h(1) = 5$ & $h'(1) = 7$. Determine $g'(5)$.

10. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems:

10.1. Find the derivative of the following function.
(i) $y = x^x$
(ii) $y = (\ln x)^{\tan 3x}$
(iii) $y = (\sqrt{x})^{\sin x}$
(iv) $y = x^{\ln x}$
(v) $y = (\cot x)^{\sin x}$

11. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems:

11.1. Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the following equation

$$f(x) + x^2 \left(f(x)\right)^3 = 10$$

and that $f(1) = 2$. Find $f'(1)$.

11.2. Find the slope of the tangent to the curve given by the equation

$$x^2 + 2xy - y^2 + x = 2$$

at point $(x, y) = (1,\ 2)$

11.3. Find $\dfrac{dy}{dx}$ given $e^{x/y} = 7x - y$.

11.4. Find the equation of the tangent line to the curve deefined by $\sqrt{y} + \sqrt{x} = 3$ at $(1, 4)$.

12. You are supposed to be able to provide an approximation of the value of a function, using the linear approximation.

Example Problems:

12.1. Find the linear approximation $L(x)$ of the function $f(x) = e^x$ at $a = 0$. Use this to estimate the value $e^{0.01}$.

12.2. Estimate the value of $\sqrt[3]{26.8}$ using a linear approximation.

13. Given the position function of a particle, you are supposed to be able to compute its velocity, acceleration, understanding its physical meaning. You should be able to determine when a particle is speeding up or down, whether it is accelerating or decelerating. You are also supposed to be able to compute the total distance travelled during the given period.

Example Problems:

13.1. Webassign HW 14 Problems 5, 6, 7, 8

13.2. The position of a particle is given by the function $s = f(t) = t^3 - 6t^2 + 9t$. Find the total distance traveled during the first 6 seconds.

13.3. A rock is thrown upward so that its height (in ft) after t seconds is given by $h(t) = 48t - 16t^2$. What is the velocity of the rock when its height is 32 ft on its way up ?

14. You are supposed to know how to provide the formulas for the trigonometric functions when the angles are given in the form of $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$.

Example Problems:

14.1. Find the formula for the following expression.
(i) $\tan\left(2\sin^{-1} x\right)$
(ii) $\cos(2\sin^{-1} x)$
(iii) $\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right)$

15. You are supposed to know the definitions of the hyperbolic functions and be able to compute their derivatives and determine their exact values.

Example Problems:

15.1. Find the exact value of the following expression.
(i) $\sinh(0)$
(ii) $\sinh(\ln 5)$
(iii) $\frac{1+\tanh(1/2)}{1-\tanh(1/2)}$.
15.2. Find the derivative of the function $f(x) = \sinh(\ln x)$, and then evaluate $f'(5)$.

16. TWO "Related Rates" problems will be given in the Final Exam. The problems are very similar to the ones given in the Webassign, and also to the examples discussed in the textbook. Of particular importance are:

- Snowball problem
- Light house problem
- Inverted circular conical tank problem (Gravel problem)
- Kite problem
- Ladder problem
- Two cars (ships) moving east-west, south-north problem
- Fred's swimming race problem
- Area (of a rectangle or a triangle) problem

17. You are supposed to be able to determine the exact values of the formulas involving the trigonometric and inverse trigonometric functions. You should pay special attention to the range of the inverse trigonometric function. You need to know the double angle formulas for sine, cosine, and tangent.

Example Problems:

17.1. Find the exact values of the following expression.

(i) $\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$
(ii) $\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$
(iii) $\sin\left(2\sin^{-1}\left(\frac{12}{13}\right)\right)$

18. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$ and the values on the critical value(s) $f(c)('s)$. You should know what the definition of a critical value is.

Example Problems:

18.1. Find the absolute maximum/minimum and local maximum/minimum of the function defined by

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on the closed interval $[-1, 4]$.

18.2. Find the absolute maximum and absolute minimum values of the function f on the given interval.
(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$
(b) $f(x) == xe^{x/2}$ on $[-3, 1]$
(c) $f(x) = x\sqrt{32 - x^2}$ on $[0, 5]$
(d) $f(t) = 2\cos t + \sin 2t$ on $[0, \pi/2]$
(e) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$

19. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

Example Problems:

19.1. The first derivative of a function f is given by

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5).$$

Find the values of x for which the function f takes

(a) local maximum, and

(b) local minimum.

19.2.

(a) Find the critical numbers of the function $f(x) = x^8(x-4)^7$.

(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?

(c) What does the First Derivative Test tell you that the Second Derivative test does not?

20. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.

Example Problems:

20.1. Compute the following limits:

(a) $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$

(b) $\lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$

(c) $\lim_{x\to 0} \dfrac{\sin x}{1-x^2}$

(d) $\lim_{x\to 0} \dfrac{7^x - 6^x}{3^x - 2^x}$

21. You are suppose to know how to compute the limits of the form $\pm\infty \times 0, \infty - \infty$.

Example Problems:

21.1. Compute the following limits:

(a) $\lim_{x\to 0^+} \sin(x)\cdot \ln(2x)$

(b) $\lim_{x\to\infty} 2x \cdot \tan\left(\dfrac{1}{3x}\right)$

(c) $\lim_{x\to 0^+} \ \cos x \cdot \arctan\left(\dfrac{3}{x}\right)$

(d) $\lim_{x\to\infty} \left(\sqrt{x^2+2x+3} - x\right)$

(e) $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

22. You are supposed to be able to compute the limits $\lim_{x\to a} [f(x)]^{g(x)}$ of the form $0^0, \infty^0, 1^\infty$.

Example Problems:

22.1. Compute the following limits:
(a) $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{7x}$
(b) $\lim_{x\to\infty} \left(\frac{2x+1}{2x-1}\right)^{4x+1}$
(c) $\lim_{x\to\infty} (2x + e^{5x})^{1/x}$
(d) $\lim_{x\to 0^+} \tan(5x)^{\sin x}$

23. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down), and also by determining the horizontal/vertical asymptotes and x-intercept (y-intercept). Go over the examples given in the Webassign and the textbook.

Example Problems:

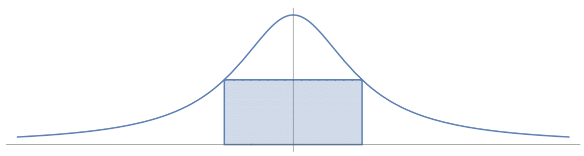
23.1. Draw the graph of the following function:
(a) $y = f(x) = \frac{x}{x^2 - 16}$
(b) $y = f(x) = \frac{x}{x^2 + 16}$
(c) $y = f(x) = \frac{1}{x^2 - 16}$
(d) $y = f(x) = \frac{x^2}{x^2 - 16}$
(e) $y = f(x) = e^{-x} \sin x$ on $[0, 2\pi]$
(f) $y = f(x) = \ln |x^2 - 10x + 24|$

24. ONE Optimization Problem will be given in the Final Exam.

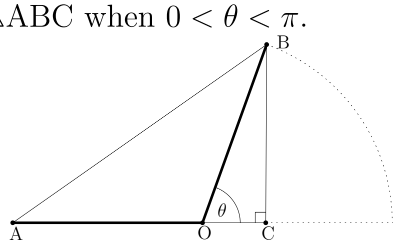
Example Problems:

- Maximize the area of a rectangle inscribed in a circle (or in an ellipse)
- Maximize the area of a rectangle inside of a right triangle
- Minimize the distance from the given point to a point on the line
- Minimize the material used to make a rectangular box (or a circular cylinder)
- Minimize the time to reach another point across the river, by first rowing and then running

24.1. What is the area of the largest rectangle that can fit in the region above the x-axis and below the curve $y = \dfrac{1}{1+x^2}$?



24.2. Two wooden bars of equal length AO = BO = 1 ft are connected by a hinge at point O so that one can rotate the bar BO around as shown in the picture below. Find the maximum area of the triangle $\triangle$ABC when $0 < \theta < \pi$.



25. You are supposed to understand that the differential equation $\dfrac{dy}{dx} = ky$ has a solution of the form $y = y(0)e^{kt}$, and should be able to apply it to analyze the population growth and radioactive decay. In the case of the radioactive decay, you should also understand the formula $m(t) = m(0)2^{-t/h}$ in terms of the half-life h.

Example Problems:

25.1. A culture of a single cell creature Amoeba is found to triple its population in three weeks. Find Its relative growth rate k.

25.2. The number of bacteria in a cell culture is initially observed to be 50. Three hours later the number is 100. Assuming that the bacteria grow exponentially, how many hours after the initial observation does the number of bacterial become equal to 700?

25.3. The half-life of cesium-137 is 30 years. Suppose we have a 60-mg sample at the beginning. How long will it take until the remain of the sample becomes 1-mg ?

25.4. A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on Earth today. Estimate the age of the parchment, knowing that the half-life of ^{14}C is 5730 years.

26. You are supposed to know how to use the Substitution Rule to compute the indefinite and/or definite integrals.

Example Problems:

26.1 Compute the following integrals:

(i) $\displaystyle\int \tan x \; dx$

(ii) $\displaystyle\int \frac{\ln x}{x} \; dx$

(iii) $\displaystyle\int_0^4 \sqrt{1+2x} \; dx$

(iv) $\displaystyle\int_0^2 x^5\sqrt{1+x^2} \; dx$

(v) $\displaystyle\int_0^{\pi/4} \sec^4 x \tan x \; dx$

(vi) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} \; dx$

(vii) $\displaystyle\int_{\pi/6}^{\pi/3} \tan x \; dx$

27. You are supposed to know how to compute the integral using the Riemann Sum. Conversely, you should know how to compute the limit in the form of Riemann sum using the integration and Fundamental Theorem of Calculus. You are also supposed to know how to compute the integral knowing its geometrical meaning.

Example Problems:

27.1. Write down the formula for approximating the integration $\int_0^1 \sqrt{1-x^2}\,dx$ as the Riemann sum dividing the interval $[0,1]$ into n equal subintervals and using the left end points.

27.2. Compute the following limits:

(i) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{3 + i \cdot \frac{5}{n}}\right) \cdot \frac{5}{n}$

(ii) $\lim_{n\to\infty} \sum_{i=1}^{3n} \left(1 + i \cdot \frac{2}{3n}\right)^5 \cdot \frac{2}{3n}$

28. You are supposed to understand the meaning of the Fundamental Theorem of Calculus, and use it to compute the derivative of a function given in the form of an integration.

Example Problems:

28.1. Look at Example 4 on Page 396 of the textbook, and Problem 13 on Page 399 and Problem 18 on Page 400.

28.2. Look at Problem 66 on Page 401.

29. You are supposed to know the basics of conic sections, and those of their **"shifted"** versions.

(1) Parabola:

- Characterization.
- Given the standard form of an equation, how to find the focus and the equation of the directrix.
- Conversely, given the focus and the equation of the directrix, how to find the standard form of an equation.

(2) Ellipse:

- Characterization.
- Given the standard form of an equation, how to find the foci and the vertices.
- Conversely, given the foci and the vertices, how to find the standard form of an equation.

(2) Hyperbola:

- Characterization.
- Given the standard form of an equation, how to find the foci, the vertices, and the equations of the asymptotes.
- Conversely, given the foci, the vertices, and/or the equations of the asymptotes, how to find the standard form of an equation.

Example Problems:

29 (1) (i) Find the vertex, focus, and directrix of the parabola given by the equation $y^2 + 4y + 4x + 8 = 0$.

(ii) Find the equation of the parabola whose focus is $(3, -9/4)$ with the directrix $y = -15/4$.

29 (2) (i) Find the vertices and foci of the ellipse defined by the equation $16x^2 - 32x + 4y^2 + 4y = 47$. What is the value of $PF_1 + PF_2$ where P is a point on the ellipse and where F_1 and F_2 are the two foci ?

(ii) Find an equation of the ellipse whose vertices are $(-2, 0)$ and $(-2, 8)$, with one of the foci being $(-2, 7)$.

29 (3) (i) Find the vertices, the foci, and the asymptotes of the hyperbola given by the equation $4x^2 - y^2 - 24x - 6y + 43 = 0$. What is the value of $|PF_1 - PF_2|$ where P is a point on the ellipse and where F_1 and F_2 are the two foci ?

(ii) Find an equation of the hyperbola whose vertices are $(-2, 5)$ and $(10, 5)$ and whose foci are $(-3, 5)$ and $(11, 5)$.