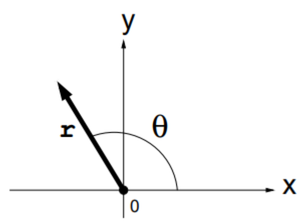
Study Guide # 1

1. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

(a) $\vec{\mathbf{v}} = \langle a, b, c\rangle = a\,\vec{\mathbf{i}} + b\,\vec{\mathbf{j}} + c\,\vec{\mathbf{k}}$; vector addition and subtraction geometrically using parallelograms spanned by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$; length or magnitude of $\vec{\mathbf{v}} = \langle a, b, c\rangle$, $|\vec{\mathbf{v}}| = \sqrt{a^2+b^2+c^2}$; directed vector from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$ given by $\vec{\mathbf{v}} = \overline{P_0P_1} = P_1 - P_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

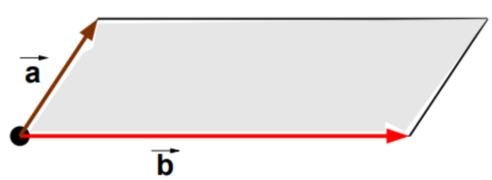
(b) Dot (or inner) product of $\vec{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3\rangle$: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3$; properties of dot product; useful identity: $\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = |\vec{\mathbf{a}}|^2$; angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$: $\cos\theta = \dfrac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|}$; $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ if and only if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$; the vector in $\mathbb{R}^2$ with length r with angle θ is $\vec{\mathbf{v}} = \langle r\cos\theta, r\sin\theta\rangle$:



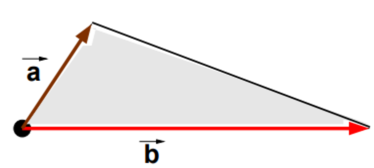
(c) Cross product (only for vectors in $\mathbb{R}^3$):

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{\mathbf{k}}$$

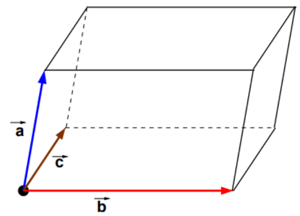
properties of cross products; $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is **perpendicular** (orthogonal or normal) to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$; area of parallelogram spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $A = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$:



the area of the triangle spanned is $A = \frac{1}{2}\,|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$:



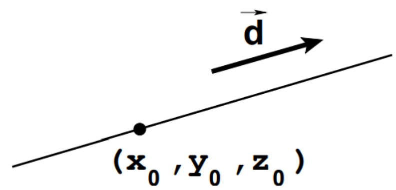
Volume of the parallelopiped spanned by $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is $V = |\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$:



2. Equation of a line L through $P_0(x_0, y_0, z_0)$ with direction vector $\vec{\mathbf{d}} = \langle a, b, c \rangle$:

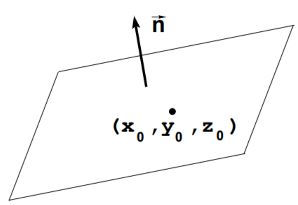
<u>**Vector Form**</u>: $\vec{\mathbf{r}}(t) = \langle x_0, y_0, z_0 \rangle + t\,\vec{\mathbf{d}}$.

<u>**Parametric Form**</u>: $\begin{cases} x = x_0 + a\,t \\ y = y_0 + b\,t \\ z = z_0 + c\,t \end{cases}$



<u>**Symmetric Form**</u>: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$. (If say $b = 0$, then $\dfrac{x - x_0}{a} = \dfrac{z - z_0}{c}$, $y = y_0$.)

3. Equation of the plane through the point $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$ ($\vec{\mathbf{n}}$ is a *normal vector* to the plane) is $\langle (x - x_0), (y - y_0), (z - z_0) \rangle \cdot \vec{\mathbf{n}} = 0$; Sketching planes (consider x, y, z intercepts).



4. Quadric surfaces (can sketch them by considering various *traces*, i.e., curves resulting from the intersection of the surface with planes $x = k$, $y = k$ and/or $z = k$); some <u>generic</u> equations have the form:

(a) *Ellipsoid*: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(b) *Elliptic Paraboloid*: $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

(c) *Hyperbolic Paraboloid (Saddle)*: $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

(d) *Cone*: $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

(e) *Hyperboloid of One Sheet*: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

(f) *Hyperboloid of Two Sheets*: $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

5. Vector-valued functions $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$; tangent vector $\vec{\mathbf{r}}\,'(t)$ for smooth curves, unit tangent vector $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|}$; principal unit normal vector $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}\,'(t)}{|\vec{\mathbf{T}}\,'(t)|}$; differentiation rules for vector functions, including:

(i) $\{\phi(t)\,\vec{\mathbf{v}}(t)\}\,' = \phi(t)\,\vec{\mathbf{v}}\,'(t) + \phi'(t)\,\vec{\mathbf{v}}(t),$ where $\phi(t)$ is a real-valued function

(ii) $(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\,' = \vec{\mathbf{u}}\cdot\vec{\mathbf{v}}\,' + \vec{\mathbf{u}}\,'\cdot\vec{\mathbf{v}}$

(iii) $(\vec{\mathbf{u}}\times\vec{\mathbf{v}})\,' = \vec{\mathbf{u}}\times\vec{\mathbf{v}}\,' + \vec{\mathbf{u}}\,'\times\vec{\mathbf{v}}$

(iv) $\{\vec{\mathbf{v}}(\phi(t))\}\,' = \phi'(t)\;\vec{\mathbf{v}}\,'(\phi(t)),$ where $\phi(t)$ is a real-valued function

6. Integrals of vector functions $\displaystyle\int \vec{\mathbf{r}}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$; arc length of curve parameterized by $\vec{\mathbf{r}}(t)$ is $L = \displaystyle\int_a^b |\vec{\mathbf{r}}\,'(t)|\,dt$; arc length function $s(t) = \displaystyle\int_a^t |\vec{\mathbf{r}}\,'(u)|\,du$; reparameterize by arc length: $\vec{\boldsymbol{\sigma}}(s) = \vec{\mathbf{r}}(t(s))$, where $t(s)$ is the inverse of the arc length function $s(t)$; the *curvature* of a curve parameterized by $\vec{\mathbf{r}}(t)$ is $\kappa = \dfrac{|\vec{\mathbf{T}}\,'(t)|}{|\vec{\mathbf{r}}\,'(t)|}$. **Note:** $\sqrt{\alpha^2} = |\alpha|$.

7. $\vec{\mathbf{r}}(t)$ = position of a particle, $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{v}}(t)$ = velocity; $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t)$ = acceleration; $|\vec{\mathbf{r}}\,'(t)| = |\vec{\mathbf{v}}(t)|$ = speed; Newton's 2^{nd} Law: $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$.

8. Domain and range of a function $f(x,y)$ and $f(x,y,z)$; *level curves* (or contour curves) of $f(x,y)$ are the curves $f(x,y) = k$; using level curves to sketch surfaces; *level surfaces* of $f(x,y,z)$ are the surfaces $f(x,y,z) = k$.

9. Limits of functions $f(x,y)$ and $f(x,y,z)$; limit of $f(x,y)$ does not exist if different approaches to (a,b) yield different limits; continuity.

10. Partial derivatives $\dfrac{\partial f}{\partial x}(x,y) = f_x(x,y) = \displaystyle\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$,

$\dfrac{\partial f}{\partial y}(x,y) = f_y(x,y) = \displaystyle\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$; higher order derivatives: $f_{xy} = \dfrac{\partial^2 f}{\partial y\,\partial x}$,

$f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$, $f_{yx} = \dfrac{\partial^2 f}{\partial x\,\partial y}$, etc; mixed partials.

11. Equation of the tangent plane to the graph of $z = f(x,y)$ at (x_0, y_0, z_0) is given by

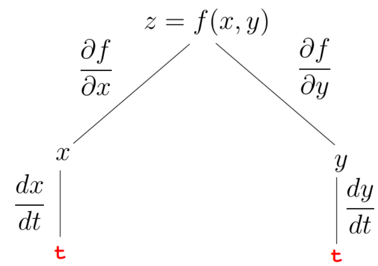
$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$.

12. Total differential for $z = f(x,y)$ is $dz = df = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy$; total differential for $w = f(x,y,z)$ is $dw = df = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy + \dfrac{\partial f}{\partial z}\,dz$; linear approximation for $z = f(x,y)$ is given by $\Delta z \approx dz$, i.e., $f(x+\Delta x,\, y+\Delta y) - f(x,y) \approx \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy$, where $\Delta x = dx,\ \Delta y = dy$;

Linearization of $f(x,y)$ at (a,b) is given by $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$; $L(x,y) \approx f(x,y)$ near (a,b).

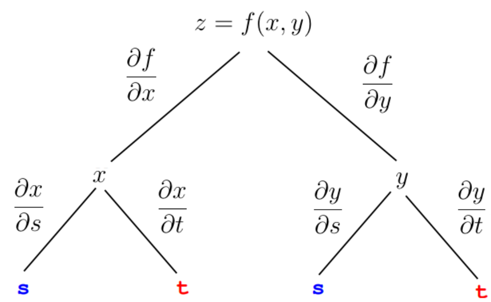
13. **<u>CHAIN RULE</u>**; different forms of the Chain Rule: Form 1, Form 2; CHAIN RULE (GENERAL FORM): Tree diagrams. For example:

(a) If $z = f(x, y)$ and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, then $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$:



(b) If $z = f(x, y)$ and $\begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases}$, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} :$$



etc.....

14. **Implicit Differentiation**:

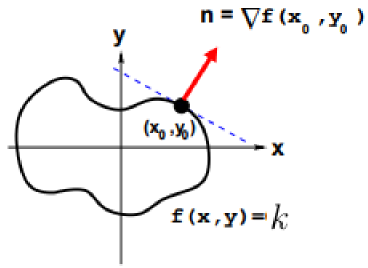
<u>Part I</u>: If $F(x, y) = 0$ defines y as function of x (i.e., $y = y(x)$), then to compute $\dfrac{dy}{dx}$, differentiate both sides of the equation $F(x, y) = 0$ w.r.t. x and solve for $\dfrac{dy}{dx}$.

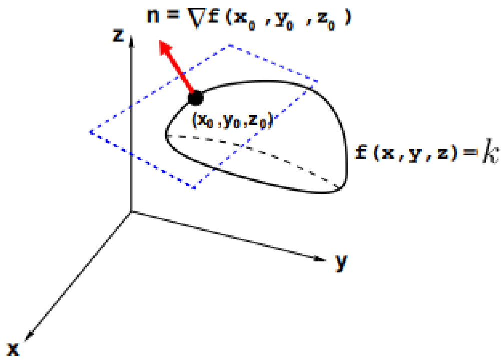
If $F(x, y, z) = 0$ defines z as function of x and y (i.e. $z = z(x, y)$), then to compute $\dfrac{\partial z}{\partial x}$, differentiate the equation $F(x, y, z) = 0$ w.r.t. x (hold y fixed) and solve for $\dfrac{\partial z}{\partial x}$. For $\dfrac{\partial z}{\partial y}$, differentiate the equation $F(x, y, z) = 0$ w.r.t. y (hold x fixed) and solve for $\dfrac{\partial z}{\partial y}$.

<u>Part II</u>: If $F(x, y) = 0$ defines y as function of x $\Longrightarrow \dfrac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$;

while if $F(x, y, z) = 0$ defines z as function of x and y $\Longrightarrow \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

15. Gradient vector for $f(x,y)$: $\nabla f(x,y) = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$, properties of gradients; gradient points in direction of maximum rate of increase of f, maximum rate of increase is $|\nabla f|$; $\nabla f(x_0, y_0) \perp$ level curve $f(x,y) = k$ and, in the case of 3 variables, $\nabla f(x_0, y_0, z_0) \perp$ level surface $f(x,y,z) = k$:





16. Directional derivative of $f(x,y)$ at (x_0, y_0) in the direction $\vec{\mathbf{u}}$: $D_{\vec{\mathbf{u}}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ must be a <u>*unit*</u> vector; tangent planes to level surfaces $f(x,y,z) = k$ (a normal vector at (x_0, y_0, z_0) is $\vec{\mathbf{n}} = \nabla f(x_0, y_0, z_0)$).

17. Relative/local extrema; critical points (points where $\nabla f = \vec{\mathbf{0}}$ or ∇f does not exist).

18. <u>2^{nd} Derivatives Test</u>: Suppose the 2^{nd} partials of $f(x,y)$ are continuous in a disk with center (a,b) and $\nabla f(a,b) = \vec{\mathbf{0}}$. Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)}$.

(a) If $D > 0$ and $f_{xx}(a,b) > 0 \implies f(a,b)$ is a local minimum value.

(b) If $D > 0$ and $f_{xx}(a,b) < 0 \implies f(a,b)$ is a local maximum value.

(c) If $D < 0 \implies f(a,b)$ is a *not* a local min or local max value. So (a,b) is a **saddle point** of f.

If $D = 0$ (or if $\nabla f(a,b)$ does not exist or f has more than 2 variables) the test gives no information and you need to do something else to determine if a relative extremum exists.