MA 26600
Study Guide # 1

(1) Special Types of First Order Equations

I. First Order Linear Equation (FOL): $\boxed{\dfrac{dy}{dt} + p(t)y = g(t)}$

Solution : $y = \dfrac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + C\right]$, where $\mu(t) = e^{\int p(t)\,dt}$

II. Separable Equation (SEP): $\boxed{\dfrac{dy}{dx} = h(x)\,g(y)}$

Solution : $\displaystyle\int \frac{1}{g(y)}\,dy = \int h(x)\,dx$

(The solution is usually given *implicitly* by the above formula. You may get additional solutions from $g(y) = 0$. You must check to see if there are extra solutions.)

III. Homogeneous Equation (HOM): $\boxed{\dfrac{dy}{dx} = f(x,y)\,,\ \text{where } f(tx,ty) = f(x,y)}$

Solution : Let $\boxed{v = \dfrac{y}{x}}$. Hence $y = xv$ and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$.

Substitute these into $\dfrac{dy}{dx} = f(x,y)$ to obtain a Separable Equation.

IV. Exact Equation (EXE): $\boxed{M(x,y)\,dx + N(x,y)\,dy = 0, \quad \text{where } \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Solution : Solution $y = \phi(x)$ given *implicitly* by $\boxed{\boldsymbol{\psi(x,y) = C}}$ where :

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = M(x,y) \quad \Longrightarrow \quad \psi = \displaystyle\int M(x,y)\,dx + h(y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\ \dfrac{\partial \psi}{\partial y} = N(x,y) \quad = \quad \dfrac{\partial \psi}{\partial y} = \dfrac{\partial}{\partial y}\left(\displaystyle\int M(x,y)\,dx + h(y)\right) \end{cases}$$

(2) Direction Fields

Direction Fields. A solution $y = \phi(t)$ to the d.e. $\dfrac{dy}{dt} = f(t,y)$ has slope $f(t,y)$ at the point (t,y). The *direction field* (or *slope field*) of the d.e. indicates the slope of solutions at various points (t, y). The direction field may be used to give qualitative information about the behavior of solutions as $t \to \infty$ (or $t \to -\infty$, or $t \to 0$, etc). Direction fields may also be used to approximate the interval where a solution through a point (t_0, y_0) is defined.

(3) Applications of 1st Order Equations.

(A1) Mixing Problems: $Q(t)$ = amount of substance in solution at time t

$$\boxed{\frac{dQ}{dt} = Rate\,In - Rate\,Out\ = r_i c_i - r_o c_o}$$

(A2) Exponential Growth/Decay: $Q(t)$ = quantity present at time t

$$\boxed{\frac{dQ}{dt} = r\,Q}$$

(A3) Newton's Law of Cooling: $T(t)$ = temperature at time t

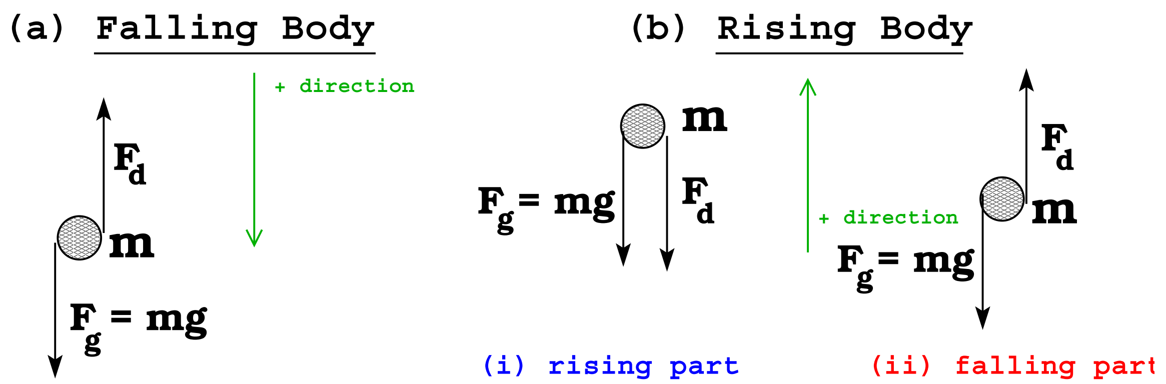
$$\boxed{\frac{dT}{dt} = k\,(T - T_a)}$$

(T_a = ambient temperature)

(A4) Falling & Rising Objects: You should be able to set up and solve simple problems using Newton's 2^{nd} Law: $F = m\,\frac{dv}{dt}$. Near the surface of the Earth, the force due to gravity is the weight of the object $F_g = mg$. Let F_d = magnitude of drag force.

(a) For falling objects, we usually let the positive direction be the *downward* direction so $\boxed{m\,\frac{dv}{dt} = mg - F_d}$.

(b) For rising objects, let the positive direction be *upward*. For the upward portion of the flight , $\boxed{m\,\frac{dv}{dt} = -mg - F_d}$; while for the downward portion of the flight, $\boxed{m\,\frac{dv}{dt} = -mg + F_d}$.



(4) Existence and Uniqueness Theorems for 1^{st} Order Equations.

(a) **THEOREM (First Order Linear)**. If $p(t)$ and $g(t)$ are continuous on an interval $\alpha < t < \beta$ containing t_0, then the IVP $\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$ has a unique solution $y = \phi(t)$ on the interval $\alpha < t < \beta$, for any y_0.

Note: The largest such open interval containing t_0 is where the solution $y = \phi(t)$ is guaranteed to exist.

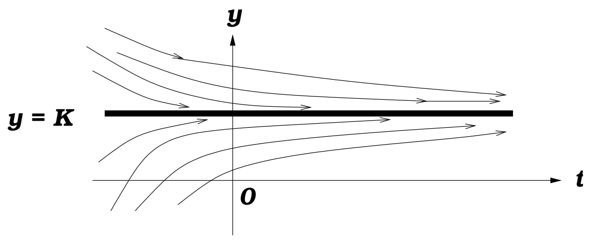
(b) **THEOREM (First Order Nonlinear)**. If $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous in some rectangle $\mathbf{R}$: $\alpha < t < \beta$, and $\gamma < y < \delta$ and (t_0, y_0) lies inside the rectangle $\mathbf{R}$, then the IVP $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$ has a unique solution on the interval $t_0 - h < t < t_0 + h$, for *some* number $h > 0$.

Note: The number h is not easy to find. The interval containing t_0 where solution exists can be estimated by looking at the direction field of the differential equation. To determine the exact interval, you must solve the IVP explicitly for y.

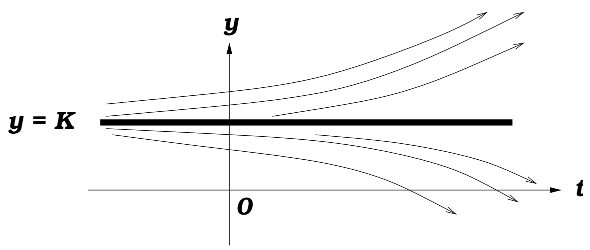
(5) Autonomous Equations: Equations of the form $\boxed{\dfrac{dy}{dt} = F(y)}$ $(*)$

are said to be *autonomous* since $\dfrac{dy}{dt}$ does not depend on the independent variable t. Such equations can have constant solutions (i.e., $y = K$) which are called *equilibrium solutions*. These solutions are found by solving $F(y) = 0$. (These are also called *critical points*.) You should be able to find all equilibrium solutions to the autonomous d.e. $(*)$ and sketch non-equilibrium solutions using the **phase line** of the differential equation $(*)$. You should also be able to classify the stability of the equilibrium solutions as follows:

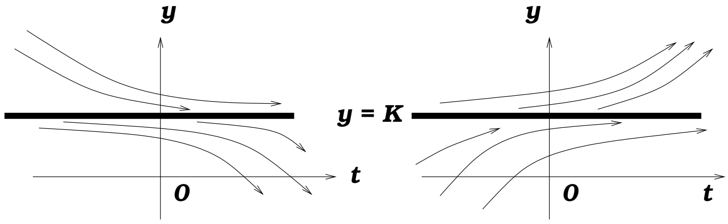
(a) **Asymptotically Stable** - Solutions which start near $y = K$ will always approach $y = K$ as $t \to \infty$:



(b) **Asymptotically Unstable** - Solutions which start near $y = K$ does not always approach $y = K$ as $t \to \infty$:



(c) **Semistable** - This is a special type of *unstable* solution. In this case solutions on one side of $y = K$ will approach $y = K$ as $t \to \infty$, while solutions on the other side of $y = K$ will approach something else:



Remark. To sketch non-equilibrium solutions of $(*)$, you do not necessarily need direction fields, you can use ordinary calculus. Since $\dfrac{dy}{dt} = F(y)$, the graph of $F(y)$ vs y will determine where the solution $y = \phi(t)$ is increasing $(F(y) > 0)$ or decreasing $(F(y) < 0)$. By the Chain Rule, $\dfrac{d^2y}{dt^2} = \dfrac{dF(y)}{dy}\,F(y)$, hence a graph of $\dfrac{dF}{dy}F$ will determine where the solution $y = \phi(t)$ is concave up $(F'F > 0)$ or concave down $(F'F < 0)$.

(6) **Euler (Tangent Line) Method.** Approximate actual solution $\phi(t)$ to $\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$

using the Euler (Tangent Line) Method :

$$\boxed{y_n = y_{n-1} + h\, f(t_{n-1}, y_{n-1})}$$

where $h =$ step size. At each iteration, $y_k \approx \phi(t_k)$, where $t_k = t_0 + hk$.

(7) **Second Order Linear Homogeneous with Equations Constant Coefficients** .

The differential equation $ay'' + by' + cy = 0$ has *Characteristic Equation* $ar^2 + br + c = 0$. Call the roots r_1 and r_2. The general solution to $ay'' + by' + cy = 0$ is as follows:

(a) If r_1, r_2 are real and distinct $\Rightarrow$ $y = C_1\, e^{r_1 t} + C_2\, e^{r_2 t}$

(b) If $r_1 = \lambda + i\mu$ (hence $r_2 = \lambda - i\mu$) $\Rightarrow$ $y = C_1\, e^{\lambda t} \cos \mu t + C_2\, e^{\lambda t} \sin \mu t$

(c) If $r_1 = r_2$ (repeated roots) $\Rightarrow$ $y = C_1\, e^{r_1 t} + C_2\, t e^{r_1 t}$

(8) **Theory of 2^{nd} Linear Order Equations**. The Wronskian is defined as

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

(a) The functions $y_1(t)$ and $y_2(t)$ are linearly independent over $a < t < b$ if $W(y_1, y_2) \neq 0$ for at least one point in the interval.

(b) **THEOREM (Existence & Uniqueness)** If $p(t), q(t)$ and $g(t)$ are continuous in an open interval $a < t < b$ containing t_0, then the IVP $\begin{cases} y'' + p(t)\, y' + q(t)\, y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$ has a unique solution $y = \phi(t)$ defined in the open interval $a < t < b$.

(c) **Superposition Principle** If $y_1(t)$ and $y_2(t)$ are solutions to the 2^{nd} order linear homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ over the interval $a < t < b$, then $y = C_1\, y_1(t) + C_2\, y_2(t)$ is also a solution for any constants C_1 and C_2.

(d) **THEOREM (Homogeneous)** If $y_1(t)$ and $y_2(t)$ are solutions to the linear homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ in some interval I and $W(y_1, y_2) \neq 0$ for some t_1 in I, then the general solution is $y_c(t) = C_1\, y_1(t) + C_2\, y_2(t)$. This is usually called the *complementary solution* and we say that $y_1(t), y_2(t)$ form a *Fundamental Set of Solutions* (FSS) to the differential equation.

(e) **THEOREM (Nonhomogeneous)** The general solution to the nonhomogeneous equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

is $y(t) = y_c(t) + y_p(t)$, where $y_c(t) = C_1\, y_1(t) + C_2\, y_2(t)$ is the general solution to the corresponding homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ and $y_p(t)$ is a particular solution to the nonhomogeneous equation $P(t)y'' + Q(t)y' + R(t)y = G(t)$.

(f) **Useful Remark** : If $y_{p_1}(t)$ is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = G_1(t)$ and if $y_{p_2}(t)$ is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = G_2(t)$, then

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$

is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = [G_1(t) + G_2(t)]$.

Practice Problems

1. Determine the order of each of these differential equations; also state whether the equation is *linear* or *nonlinear*:
(a) $yy' + x = 1$ (b) $xy' + y = 1$ (c) $(y')^3 + ty = 1$ (d) $y''' + \sqrt{t}\, y = 1$

2. (a) Which of the functions $y_1(t) = t$ and $y_2(t) = -t$ is/are solutions of the IVP $yy' = t,\ y(0) = 0$?
(b) Which of the functions $y_1(t) = t$ and $y_2(t) = -t$ are is/solutions of the IVP $yy' = t,\ y(1) = 1$?

3. For what value(s) of r is $y = e^{rx}$ a solution of $y'' - 5y' + 6y = 0$?

4. (a) Show that $y = x^3$ is a solution of the initial value problem $y' = 3y^{2/3},\ y(0) = 0$.
(b) Find a different solution of the initial value problem.

5. Find an explicit solution of the initial value problem $x^2 y' = y^2, \quad y(1) = \dfrac{1}{2}$. Indicate the interval in which the solution is valid.

6. (a) Find an implicit solution of the initial value problem $y' = \dfrac{2x}{2y+1}, \quad y(0) = 0$.
(b) Find an explicit solution of the initial value problem $y' = \dfrac{2x}{2y+1}, \quad y(0) = 0$.

7. For what value(s) of a is the solution of the IVP $y' - y + 2e^{-t} = 0,\ y(0) = a$ bounded on the interval $t \geq 0$?

8. Determine whether each of the following differential equations is linear, separable, homogeneous, and/or exact or none of these. (a) $2x + y + (x + 3y)\dfrac{dy}{dx} = 0$ (b) $x + 3y + (2x + y)\dfrac{dy}{dx} = 0$

(c) $(x+3y+1)dx+(2x+y+1)dy=0$ (d) $2xy+1+(x^2+1)\dfrac{dy}{dx}=0$

(e) $(y^2+1)dy+(x^2+1)dx=0$

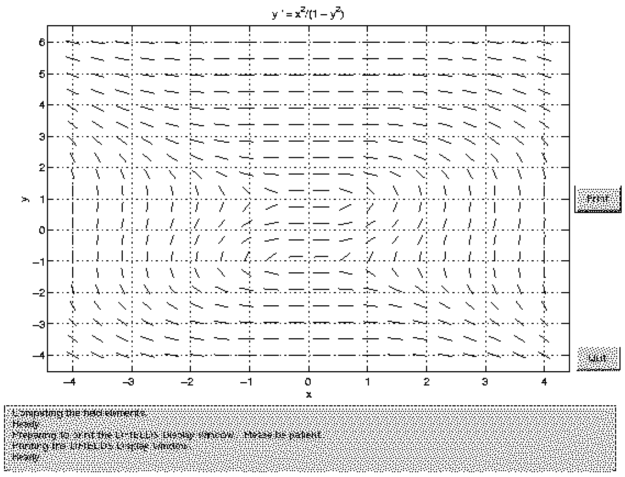
9. Find implicit solutions to (a) $x^2+y^2-2xyy'=0$ (b) $(1+y^2)\,dx-2xy\,dy=0$
(c) $x+y^2+2xyy'=0$

10. Find an implicit form of the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{x^2+y^2}{xy}$.

11. Find an implicit solution of the IVP $2xy+1+(x^2+2y)\dfrac{dy}{dx}=0,\quad y(1)=-1.$

12. If $xy'+(x+1)y=2xe^{-x}$ and $y(1)=0$, then $y(2)=$?

13. Use the given direction field to sketch the solution of the corresponding initial value problem $y'=f(t,y)$, $y(t_0)=y_0$ for the indicated initial value (t_0,y_0) :
(a) $(0,0)$ (b) $(0,2)$ (c) $(-1,3)$ (d) $(0,4)$



14. For each of the initial value problems determine the largest interval for which a unique solution is guaranteed :

(a) $y'-\dfrac{2}{t}y=\dfrac{1}{t},\ y(1)=0$ (b) $y'+(\tan t)y=\sec t,\ y(0)=0$

(c) $y'+\dfrac{x}{x^2-9}y=\dfrac{1}{x-2},\ y(0)=1$ (d) $(x+4)y'-xy=\dfrac{1}{x},\ y(-2)=1$

15. For each of the initial value problems determine all initial points (t_0,y_0) for which a unique solution is guaranteed in some interval $t_0-h<t<t_0+h$:

(a) $y'=t^2+y^2,\ y(t_0)=y_0$ (b) $y'=t/y,\ y(t_0)=y_0$ (c) $y'=\sqrt{t^2+y^2},\ y(t_0)=y_0$

(d) $y'=t^{1/3}+y^{1/3},\ y(t_0)=y_0$ (e) $y'=\dfrac{\sqrt{1-y^2}}{t-2},\ y(t_0)=y_0$

16. Find the explicit solution of the initial value problem $y'=y^2-1$, $y(0)=-2$. Where is this solution defined ?

17. Suppose y' is proportional to y, $y(0) = 4$, and $y(2) = 2$. Set up and solve an initial value problem that gives y in terms of t. For what value of t does $y(t) = 3$?

18. A thermometer reads 36° when it is moved into a 70° room. Five minutes later the thermometer reads 50°. Set up and solve an initial value problem that gives the thermometer reading t minutes after it is moved into the room. What will it read ten minutes after it is moved into the room?

19. At time $t = 0$ a 500 gallon tank contains 40 pounds of salt mixed in 100 gallons of water. A solution that contains 3 lb of salt per gallon of solution is then pumped into the tank at a constant rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 3 gal/min. Set up and solve an initial value problem that gives the amount of salt in the tank after t minutes. What is the concentration of salt in the tank at the time the tank becomes full?

20. A huge 300 gallon radiator is full of a 60% antifreeze solution. Pure water is poured in at a rate of 5 gal/min and the stirred mixture is drained at the same rate. How long do we pour water into the radiator to get a 50% antifreeze solution ?

21. Set up and solve an initial value problem that gives the vertical velocity of a 128-lb parachutist t seconds after she jumps from an airplane that is flying horizontally at an altitude of 5000 feet. Assume that air resistance is eight times the speed and ignore horizontal motion and downward direction is positive.

22. Consider the differential equation $\dfrac{dy}{dt} = y(y-2)$.

(a) What are the equilibrium solutions?

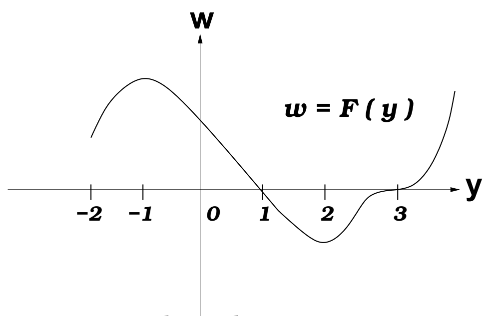
(b) Which equilibrium solutions are stable/unstable?

(c) Sketch the graph of the solution of the differential equation for $t \geq 0$ with each of the initial values $y(0) = -2/3,\ y(0) = 0,\ y(0) = 2/3,\ y(0) = 4/3,\ y(0) = 2,\ y(0) = 8/3$.

(d) Find the explicit solution of the initial value problem $\dfrac{dy}{dt} = y(y-2), y(0) = y_0$.

(e) For what values of t is the solution in (d) valid?

23. Consider the differential equation $\dfrac{dy}{dt} = F(y)$, where the graph of $F(y)$ is indicated below.



(a) What are the equlibrium solutions?

(b) Which equilibrium solutions are stable?

(c) Sketch some solutions to $\dfrac{dy}{dt} = F(y)$.

24. Estimate the solution at $t = 1.5$ to the IVP $y' = 2t - 5y$, $y(1) = -2$ using the Euler Method with $h = 0.25$. What is the true solution at $t = 1.5$?

25. Find the general solution to (a) $y'' - 4y' + 4y = 0$ (b) $y'' + 4y' + 5y = 0$.

26. For what value of α will the solution to the IVP $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha \\ y'(0) = 2 \end{cases}$ satisfy $y \to 0$ as $t \to \infty$?

27. Find the largest open interval guaranteed by the Existence and Uniqueness Theorem for which the initial value problem $3x^2y'' + y' + \dfrac{1}{x-2}y = \dfrac{1}{x-3}$, $y(1) = 3,\ y'(1) = 2$, has a unique solution.

<u>ANSWERS</u>

(1) (a) 1^{st} order nonlinear (b) 1^{st} order linear (c) 1^{st} order nonlinear (d) 3^{rd} order linear

(2) (a) y_1 and y_2 (b) y_1 only **(3)** $r = 2$, $r = 3$

(4) (a) $y' = 3x^2 = 3(x^3)^{2/3} = 3y^{2/3}\,; 0 = 3(0)^{2/3}$ (b) $y \equiv 0$ **(5)** $y = \dfrac{x}{x+1}$, $x > -1$

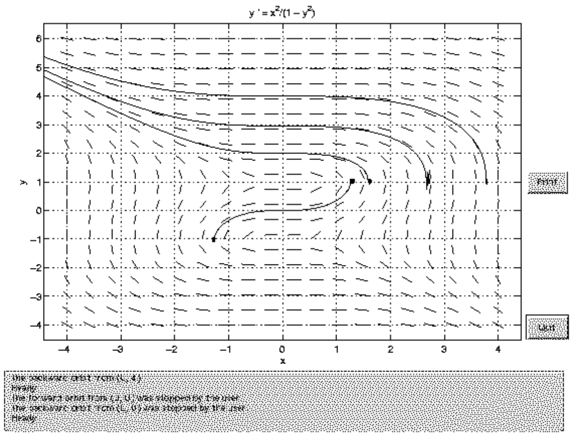
(6) (a) $y^2 + y = x^2$ (b) $y = \dfrac{-1 + \sqrt{4x^2+1}}{2}$ **(7)** $a = 1$ **(8)** (a) HOME and EXE (b) HOME

(c) none of these types (d) FOL and EXE (e) SEP and EXE

(9) (a) (HOME) $-\ln|1 - (\frac{y}{x})^2| = \ln|x| + C$ and $y = x$ and $y = -x$ (b) (SEP) $y^2 + 1 = Cx$

(c) (EXE) $x^2 + 2xy^2 = C$ **(10)** $\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \ln|x| + C$ **(11)** $x^2y + x + y^2 = 1$

(12) $y(2) = \frac{3}{2}e^{-2}$ **(13)** See below :



(14) (a) $t > 0$ (b) $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$ (c) $-3 < x < 2$ (d) $-4 < x < 0$

(15) (a) all (t_0, y_0) (b) all (t_0, y_0) with $y_0 \neq 0$ (c) all $(t_0, y_0) \neq (0, 0)$ (d) all (t_0, y_0) with $y_0 \neq 0$ (e) all (t_0, y_0) where $-1 < y_0 < 1$ and $t_0 \neq 2$

(16) $y = \dfrac{1 + 3e^{2x}}{1 - 3e^{2x}}$, solution defined for $-\frac{1}{2}\ln 3 < x < \infty$.

(17) $\begin{cases} y' = ky \\ y(0) = 4 \\ y(2) = 2 \end{cases}$; $y = 4e^{(\ln 0.5)t/2}$, $t = \dfrac{2\ln 0.75}{\ln 0.5} \approx 0.83$

(18) $\begin{cases} T' = k(T - 70) \\ T(0) = 36 \\ T(5) = 50 \end{cases}$; $T = 70 - 34e^{(\ln(10/17))t/5}$, $T(10) \approx 58.2°$

(19) $\begin{cases} Q' = 15 - \dfrac{3Q}{100 + 2t} \\ Q(0) = 40 \end{cases}$; $Q = 3(100+2t) - 260{,}000(100+2t)^{-3/2}$, $\dfrac{Q(200)}{500} \approx 2.95$ lbs/gal

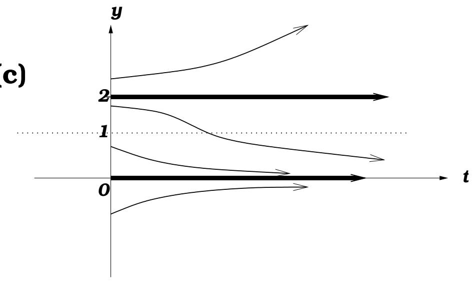
(20) If $Q(t) = \#$ gals of antifreeze, then $Q' = -\frac{Q}{60}$, $Q(0) = 180$ and so $Q(t) = 180e^{-\frac{t}{60}}$. newline Hence $t = -60 \ln \frac{5}{6} \approx 10.94$ minutes

(21) $\begin{cases} 4\dfrac{dv}{dt} = 128 - 8v \\ \\ v(0) = 0 \end{cases}$; $v = 16(1 - e^{-2t})$

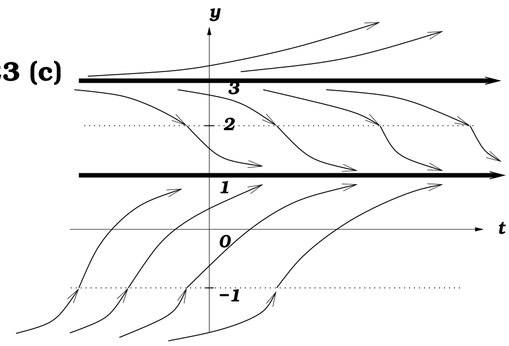
(22) (a) $y = 0$ and $y = 2$ (b) $y = 0$ is stable, $y = 2$ is unstable (c) **See below**
(d) $y = \dfrac{2y_0}{y_0 - (y_0 - 2)e^{2t}}$ (e) The solution is valid for all t if $0 \leq y_0 \leq 2$. If $y_0 > 2$ or $y_0 < 0$, the solution is valid only for $-\infty < t < \dfrac{1}{2} \ln \left(\dfrac{y_0}{y_0 - 2} \right)$.

(23) (a) $y = 1$ and $y = 3$ (b) only $y = 1$ is stable (c) **See below**
(24) $y_2 = 0.375$, true solution $\phi(1.5) = \frac{1}{25}(13 - 58\, e^{-2.5}) \approx 0.3296$

22 (c)



23 (c)



(25) (a) $y = C_1 e^{2t} + C_2 t e^{2t}$ (b) $y = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$ **(26)** $\alpha = -2$ **(27)** $0 < x < 2$