TEC Visual 12.3B shows how Figure 4 changes when we vary **a** and **b**.

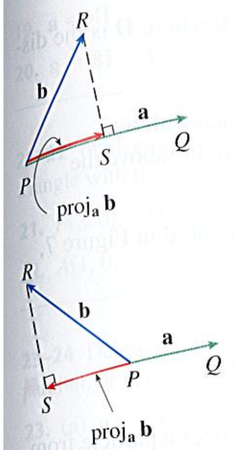


FIGURE 4
Vector projections

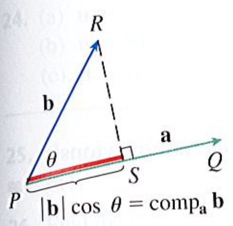


FIGURE 5
Scalar projection

Projections

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of **b**).

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**. (See Figure 5.) This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\qquad \text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\qquad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$ ■

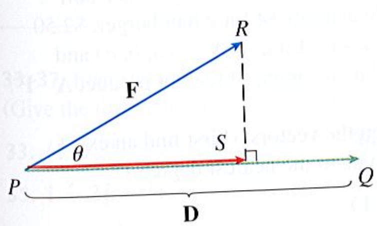


FIGURE 6

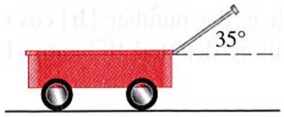
One use of projections occurs in physics in calculating work. In Section 6.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along **D** and the distance moved:

$$W = (|\mathbf{F}|\cos\theta)|\mathbf{D}|$$

But then, from Theorem 3, we have

$$\boxed{12} \qquad W = |\mathbf{F}||\mathbf{D}| \cos\theta = \mathbf{F}\cdot\mathbf{D}$$

Thus the work done by a constant force **F** is the dot product $\mathbf{F}\cdot\mathbf{D}$, where **D** is the displacement vector.



EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

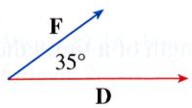


FIGURE 7

SOLUTION If **F** and **D** are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 35^\circ$$

$$= (70)(100) \cos 35^\circ \approx 5734 \text{ N}\cdot\text{m} = 5734 \text{ J}$$

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2\rangle$, so by Equation 12, the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = \langle 3, 4, 5\rangle \cdot \langle 2, 5, 2\rangle$$

$$= 6 + 20 + 10 = 36$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

12.3 EXERCISES

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
   (a) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ (b) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
   (c) $|\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ (d) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$
   (e) $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ (f) $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$

2–10 Find $\mathbf{a}\cdot\mathbf{b}$.

2. $\mathbf{a} = \langle 5, -2\rangle$, $\mathbf{b} = \langle 3, 4\rangle$

3. $\mathbf{a} = \langle 1.5, 0.4\rangle$, $\mathbf{b} = \langle -4, 6\rangle$

4. $\mathbf{a} = \langle 6, -2, 3\rangle$, $\mathbf{b} = \langle 2, 5, -1\rangle$

5. $\mathbf{a} = \langle 4, 1, \frac{1}{4}\rangle$, $\mathbf{b} = \langle 6, -3, -8\rangle$

6. $\mathbf{a} = \langle p, -p, 2p\rangle$, $\mathbf{b} = \langle 2q, q, -q\rangle$

7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

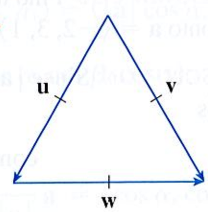
8. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

9. $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$, the angle between **a** and **b** is 30°

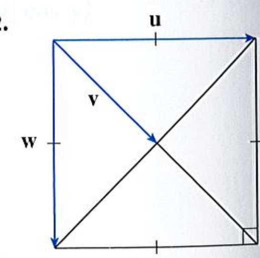
10. $|\mathbf{a}| = 80$, $|\mathbf{b}| = 50$, the angle between **a** and **b** is $3\pi/4$

11–12 If **u** is a unit vector, find $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\cdot\mathbf{w}$.

11.



12.



13. (a) Show that $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.
    (b) Show that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$.

14. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$4 for a hamburger, \$2.50 for a hot dog, and \$1 for a soft drink. If $\mathbf{A} = \langle a, b, c\rangle$ and $\mathbf{P} = \langle 4, 2.5, 1\rangle$, what is the meaning of the dot product $\mathbf{A}\cdot\mathbf{P}$?

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15. $\mathbf{a} = \langle 4, 3\rangle$, $\mathbf{b} = \langle 2, -1\rangle$

16. $\mathbf{a} = \langle -2, 5\rangle$, $\mathbf{b} = \langle 5, 12\rangle$