Strategy for Trigonometric Integration

TYPES

1. $\int \sin^m x \cos^n x dx$
2. $\int \tan^m x \sec^n x dx$
3. $\begin{cases} \int \sin mx \cos nx dx \\ \int \sin mx \sin nx dx \\ \int \cos mx \cos nx dx \end{cases}$

1. **Case:** m **odd or** n **odd**

Attitude: Smile ! (Easy !)

Strategy (Say, n is odd. The case where m is odd is identical. If both m and n are odd, you can go either way.): Extract one $\cos x$ to see $\cos x dx$ at the end. Then use the substitution $u = \sin x$ via the equality $\cos^2 x = 1 - \sin^2 x$.

Example:

$$\begin{aligned}
\int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cdot \cos x dx \\
&= \int \sin^4 x (\cos^2 x)^2 \cdot \cos x dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cdot \cos x dx \\
& \quad \{(\text{Substitution})\ u = \sin x\ \ \&\ \ du = \cos x dx\} \\
&= \int u^4 (1 - u^2)^2 \cdot du \\
&= \int (u^4 - 2u^6 + u^8) du \\
&= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \\
&= \frac{\sin^5 x}{5} - \frac{2 \sin^7 x}{7} + \frac{\sin^9 x}{9} + C
\end{aligned}$$

Case: m **even and** n **even**

Attitude: Not smile but not frown, either. (Not so good but not so bad, either.)

Strategy: Using the double angle formula

$$\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}$$

drop the degree.

Example:

$$\begin{aligned} \int \sin^2 x \cos^2 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\ &= \frac{1}{4}\int \left(1-\cos^2 2x\right) dx \\ &= \frac{1}{4}\int \left(1-\frac{1+\cos 4x}{2}\right) dx \\ &= \frac{1}{8}\int (1-\cos 4x)\, dx \\ &= \frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C \\ &= \frac{1}{8}x-\frac{1}{32}\sin 4x+C \end{aligned}$$

2. **Case: n even,** $n \neq 0$

Attitude: Smile ! (Easy !)

Strategy: Extract $\sec^2 x$ to see $\sec^2 x dx$ at the end. Then use the substitution $u = \tan x$ via the equality $\sec^2 x = 1 + \tan^2 x$ to turn the remaining even powers of $\sec x$ into the expression in terms of $\tan x$.

Example:

$$\begin{aligned} \int \tan^3 x \sec^4 x dx &= \int \tan^3 x \sec^2 x \cdot \sec^2 x dx \\ &= \int \tan^3 x (1 + \tan^2 x) \cdot \sec^2 x dx \\ &\quad \left\{ (\text{Substitution})\ u = \tan x \ \ \&\ \ du = \sec^2 x dx \right\} \\ &= \int u^3 (1 + u^2) du \\ &= \int (u^3 + u^5) du \\ &= \frac{u^4}{4} + \frac{u^6}{6} + C \end{aligned}$$

Case: $n = 0$

Attitude: Not smile but not frown, either. (Not so good but not so bad, either.)

$\boxed{m = 1}$

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &\quad \{ (\text{Substitution})\ u = \cos x \ \ \&\ \ du = -\sin x dx \} \\ &= \int \frac{1}{u} (-du) \\ &= -\int \frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C \\ &= \ln |\cos x|^{-1} + C \\ &= \ln |\sec x| + C \end{aligned}$$

$\boxed{m > 1}$

Strategy: Drop the number m using the equality $\tan^2 x = \sec^2 x - 1$.

Examples:

$$
\begin{aligned}
m = 3 : \int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x (\sec^2 x - 1) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx
\end{aligned}
$$

We already know how to compute $\int \tan x dx$.

We can compute $\int \tan x \sec^2 x dx$ by using integration by substitution $u = \tan x$ and $du = \sec^2 x dx$

$$
\begin{aligned}
\int \tan x \sec^2 x dx &= \int u du \\
&= \frac{u^2}{2} + C \\
&= \frac{\tan^2 x}{2} + C
\end{aligned}
$$

$$
\begin{aligned}
m = 5 : \int \tan^5 x dx &= \int \tan^3 x \tan^2 x dx \\
&= \int \tan^3 x (\sec^2 x - 1) dx \\
&= \int \tan^3 x \sec^2 x dx - \int \tan^3 x dx
\end{aligned}
$$

We already know how to compute $\int \tan^3 x dx$.

We can compute $\int \tan^3 x \sec^2 x dx$ by using integration by substitution $u = \tan x$ and $du = \sec^2 x dx$

$$\begin{aligned}\int \tan^3 x \sec^2 x dx &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{\tan^4 x}{4} + C\end{aligned}$$

Case: m **odd** & n **odd**

Attitude: Smile ! (Easy !)

Strategy: Extract $\tan^x \sec x$ to see $\tan x \sec x dx$ at the end. Then use the substitution $u = \sec x$ via the equality $\tan^2 x = \sec^2 x - 1$ to turn the remaining even powers of $\tan x$ into the expression in terms of $\sec x$.

Example:

$$\begin{aligned}\int \tan^3 x \sec^5 x dx &= \int \tan^2 x \sec^4 x \cdot \tan x \sec x dx \\ &= \int (\sec^2 x - 1) \sec^4 x \cdot \tan x \sec x dx \\ &\quad \{(\text{Substitution})\ u = \sec x \ \ \&\ \ du = \tan x \sec x dx\} \\ &= \int (u^2 - 1) u^4 du \\ &= \int (u^6 - u^4) du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C\end{aligned}$$

Case: m **even &** n **odd**

Attitude: Hard ! (Frowning !)

Note: By using the equality $\tan^2 x = \sec^2 x - 1$, one can express $\tan^m x$ part only in terms of (even powers of) $\sec x$. Therefore, the problem is reduced to computing $\int \sec^n x dx$ with n being odd (the case of n being even is already discussed).

$\boxed{n = 1}$

$$\begin{aligned} \int \sec x dx &= \int \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)} dx \\ &= \ln|\sec x + \tan x| + C \end{aligned}$$

$\boxed{n > 1 \ \& \ n \text{ odd}}$

Strategy: Using integration by parts, drop the numebr n.

Example:

$$\begin{aligned} \int \sec^3 x dx &= \int u dv \\ \text{(Substitution)} & \quad \begin{cases} u = \sec x & \& \ v = \tan x \\ du = \sec x \tan x dx & \& \ dv = \sec^2 x dx \end{cases} \\ &= uv - \int v du \\ &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx - \int \sec x dx \end{aligned}$$

Therefore, moving $-\int \sec^3 x dx$ on the right-hand side to the left-hand side, we conclude

$$2\int \sec^3 x dx = \sec x \tan x + \int \sec x dx.$$

That is to say,

$$\int \sec^3 x dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x dx\right),$$

where we already know how to compute $\int \sec x dx$.

$$\begin{aligned}
\int \sec^5 x dx &= \int u dv \\
\text{(Substitution)} & \quad \begin{cases} u = \sec^3 x & \& \ v = \tan x \\ du = 3\sec^3 x \tan x dx & \& \ dv = \sec^2 x dx \end{cases} \\
&= uv - \int v du \\
&= \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x dx \\
&= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1) dx \\
&= \sec^3 x \tan x - 3\int \sec^5 x dx + 3\int \sec^3 x dx
\end{aligned}$$

Therefore, moving $-3\int \sec^5 x dx$ on the right-hand side to the left-hand side, we conclude

$$4\int \sec^5 x dx = \sec^3 x \tan x + 3\int \sec^3 x dx.$$

That is to say,

$$\int \sec^5 x dx = \frac{1}{4}\left(\sec^3 x \tan x + 3\int \sec^3 x dx\right),$$

where we already know how to compute $\int \sec^3 x dx$.

3. Attitude: Smile ! (Easy !)

Strategy: Using the formulas

$$\begin{cases} \sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] \\ \sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] \\ \cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] \end{cases}$$

we split the multiplication of sin and cos into the addition.

Example:

$$\begin{aligned} \int \sin 4x \cos 5x dx &= \int \frac{1}{2}\left[\sin(4-5)x + \sin(4+5)x\right] dx \\ &= \frac{1}{2}\int \left[\sin(-x) + \sin 9x\right] dx \\ &= \frac{1}{2}\int \left[-\sin x + \sin 9x\right] dx \\ &= \frac{1}{2}\left[\cos x - \frac{1}{9}\cos 9x\right] + C \\ &= \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C \end{aligned}$$