Study Guide for Exam 1

1. You are supposed to know the basics of the description of the geometric objects in 2-dimensional and 3-dimensional coordinate systems.

• You are supposed to be able to determine the center and radius of a sphere by "completing the square", given the equation of the form

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0.$$

• You are also suppoed to be able to compute the distance between two given points.

• You are supposed to be able to describe the region defined by some inequalities.

Example Poblems:

1.1. Compute the distance from the point $P = (2, 1, -5)$ to the closest point on the sphere defined by the equation

$$x^2 + y^2 + z^2 + 2x - 6y + 9 = 0,$$

and the distance to the farthest point on the same sphere.

1.2. Write inequalities which describe the geometric objects below.

(i) The solid cylinder whose central axis is the line given by the equations $x = 3, y = -5$, and the cross section perpendicular to the axis is a disk of radius 2.

(ii) The solid upper hemisphere of the sphere of radius 7 centered at $(1, -2, 3)$. (We choose the z-axis to be the verical one, while the xy-plane is horizontal. The word "upper" is with respect to the vertical z-axis.)

2. You are supposed to be able to carry out the basic operations among the vectors, addition, subtraction, scalar multiplication, and understand the geometric meaning of each operation.

Example Poblems:

2.1. Find a unit vector whose direction is the same as the vector $\overrightarrow{PQ}$ where P and Q are the following two points in the 3-space

$$\begin{aligned} P &= (3, -1, 4) \\ Q &= (7, 2, -5) \end{aligned}$$

2.2. Let $\vec{u} = \langle 1, 2 \rangle, \vec{v} = \langle 3, -4 \rangle$. Find α, β so that $\vec{w} = \langle 1, 0 \rangle = \alpha\vec{u} + \beta\vec{v}$.

3. You are supposed to be able to compute the dot product $\vec{a}\cdot\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$.

• You are supposed to understand the geometrical interpretation of the dot product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

• You should be able to use the orthogonality criterion in terms of the dot product

$$\vec{a} \perp \vec{b} \Longleftrightarrow \vec{a}\cdot\vec{b} = 0.$$

Example Poblems:

3.1. Determine the numbers α and β so that $\vec{a} = \langle \frac{1}{3}, \frac{1}{3}, \alpha \rangle$ is a unit vector and that $\vec{a}$ is orthogonal to $\vec{b} = \langle \beta, 0, \sqrt{2} \rangle$.

3.2. Determine the angle between the following two tangent lines: one is to the curve $y = 3x$ at point $(1,3)$ and the other is to the curve $y = 3x^5$ at point $(1,3)$.

4. You are supposed to be able to compute the cross product $\vec{a}\times\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$.

• You are supposed to understand the geometrical characterization of the cross product $\vec{a}\times\vec{b}$ as the vector orthogonal to both $\vec{a}$ and $\vec{b}$, where the direction is determined by the right hand rule, with the magnitude being equal to the area of the parallelogram fromed by the two vectors $\vec{a}$ and $\vec{b}$.

• As another application of the characterization above,

• As an application of the characterization above, you should be able to use the criterion for two vectors to be parallel in terms of the cross product

$$\vec{a} \parallel \vec{b} \Longleftrightarrow \vec{a}\times\vec{b} = \vec{0}.$$

• You should be able to compute the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ as the determinant of 3×3 matrix formed by the vectors $\vec{a},\vec{b},\vec{c}$. You are also supposed to know its geometrical interpretation as the volume of the parallelepiped formed by the the three vectors.

Example Poblems:

4.1. Find the cross product $\vec{a}\times\vec{b}$ where

$$\begin{aligned} \vec{a} &= 2\vec{i} + \vec{j} - \vec{k}, \\ \vec{b} &= \vec{i} + 3\vec{j} + 2\vec{k}. \end{aligned}$$

4.2. Let $\vec{a} = \langle 3, -1, 5 \rangle$. Find a vector $\vec{v}$ such that

(i) $\vec{v}$ is perpendicular to $\vec{a}$, and

(ii) $\vec{a}\times\vec{v} = \langle -1, 2, 1 \rangle$.

4.3. Determine the constant k so that the following three vectors are coplanar

$$\begin{aligned}\vec{a} &= \langle 1, 4, k\rangle,\\ \vec{b} &= \langle 2, -1, 4\rangle,\\ \vec{c} &= \langle 0, -9, 18\rangle.\end{aligned}$$

4.4. Find the area of the triangle for med by the following three points

$$\begin{aligned}&P(1, 0, 1),\\ &Q(-2, 1, 3),\\ &R(4, 2, 5).\end{aligned}$$

5. You are supposed to be able to compute the vector projection $\mathbf{proj}_{\vec{a}}\vec{b}$ of a vector $\vec{b}$ onto $\vec{a}$, and scalar projection $\mathbf{comp}_{\vec{a}}\vec{b}$ by the formulas

$$\begin{cases} \mathbf{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a} \\ \mathbf{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\sqrt{\vec{a}\cdot\vec{a}}} \end{cases}$$

WARNING: Make a clear distiction between $\mathbf{proj}_{\vec{a}}\vec{b}$ and $\mathbf{proj}_{\vec{b}}\vec{a}$.

Example Poblems:

5.1. Find the scalar projection and vector projection of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

6. You are supposed to be able to compute the direction angles and direction cosines.

Example Poblems:

6.1. Find the direction angles of the vector $\vec{a} = \langle 2, -1, 3\rangle$ by computing its direction cosines.

6.2. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

7. You are suppoesed to be able to compute the area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ by the formula

$$\int_a^b |f(x) - g(x)|dx.$$

Example Poblems: Look at the problems in Webassign HW 5.

8. You are supposed to be able to compute the volume of a solid obtained by rotating the region enclosed by some curves around a fixed axis , using

(i) **the washer method**, and

(ii) **the method of cylindrical shells**.

Example Poblems:

8.1. Write down the formulas to compute the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2$ and $x = y^2$ around $y = -3$, using

(i) the washer method, and

(ii) the method of cylindrical shells.

8.2. Compute the volume of a northern cap of a sphere with radius r and height h.

8.3. Find the volume of the solid obtained by rotating around $y = 2$ the region enclosed by $y = 1$, $y = e^{-x}$, $x = 2$ using the washer method.

8.4. Find the volume of the solid obtained by rotating around y-axis the region bounded by the curves $y = 3 + 2x - x^2$ and $x + y = 3$ using the method of cylindrical shells.

8.5. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sin x$, $y = 0$, $x = \pi/2$.

9. You are supposed to be able to compute the volume of a solid, given the description of its base and its cross sections.

Example Poblems:

9.1. Find the volume of the solid S described below:

The base of S is a circular disk of radius 1. The cross sections perpendicular to the base and x-axis are isosceles right triangles with hypotenuse on the base.

9.2. Look at Example 9 on Page 446 of the textbook.

10. You are suppoesed to be able to compute the amount of work needed to carry out a task.

Typical examples are:

- work needed to empty the water from a tank in the shape of an inverted circulat cone (Look at Example 5 in 6.4 on Page 457 of the textbook.),
- work needed to stretch a spring (Look at Example 3 in 6.4 on Page 457 of the textbook and Problem 4 in Webassign HW 8.),
- work needed to lift a chain (Look at Problem 19 in 6.4 on Page 459).

11. You are suppoesed to be able to compute the average value f_{ave} of a function $y = f(x)$ on the interval $[a, b]$ by the formula

$$f_{\text{ave}} = \frac{\displaystyle\int_a^b f(x)dx}{b-a}.$$

Example Poblems:

11.1. Find the number b so that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

11.2. Compute the average value of the function $f(x) = \sin^4 x \cos x$ over the interval $[0, \pi]$.

12. You are suppoed to be able to evaluate the integral using integration by parts.

Example Poblems:

(i) $\displaystyle\int \ln(x)\ dx$

(ii) $\displaystyle\int xe^x\ dx$

(iii) $\displaystyle\int x \sin x\ dx$

(iv) $\displaystyle\int x \cos(\pi x)\ dx$

(v) $\displaystyle\int e^x \sin x\ dx$

(vi) $\displaystyle\int_0^1 \tan^{-1} x\ dx$