Study Guide for Exam 3

1. You are supposed to know that a geometric series with

- $A(\neq 0)$: the intial term, and
- r: the ratio of the adjacent terms so that $a_{n+1} = a_n \cdot r$

$$\begin{cases} \text{converges to } \dfrac{A}{1-r} & \text{when } |r| < 1, \\ \text{diverges} & \text{when } |r| \geq 1. \end{cases}$$

Warning: Let's look at the case where $|r| < 1$ and hence the geometric series converges. When you write the general term of the geometric series as

$$a_n = a \cdot r^{n-1}$$

and when the index starts with $n = 1$, we compute the initial term to be

$$A = a \cdot r^{1-1} = a$$

and hence

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{A}{1-r} = \frac{a}{1-r}.$$

However, when the index starts with $n = n_o \neq 1$, then the initial term is NOT equal to a and hence we have

$$\sum_{n=n_o}^{\infty} a \cdot r^{n-1} = \frac{A}{1-r} = \frac{a \cdot r^{n_o - 1}}{1-r} \neq \frac{a}{1-r}.$$

Example Problems

1.1. Judge whether the following geometric series converges or diverges. When it converges, compute its value.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{7^n}$ (2) $\displaystyle\sum_{n=3}^{\infty} \frac{3^{n+1}}{7^n}$

(3) $\displaystyle\sum_{n=5}^{\infty} \frac{7^{n+1}}{5^n}$ (4) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1} - 5^{n-2}}{7^n}$

(5) $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+1} - 5^{n-2}}{7^n}$ (6) $\displaystyle\sum_{n=3}^{\infty} \frac{3^{3n+1} - 5^{2n-2}}{31^n}$

(7) $\displaystyle\sum_{n=2}^{\infty} \frac{3^{3n+1} - 5^{3n-2}}{7^{2n}}$

2. You are supposed to know how to detemine whether a given series is convergent or divergent, using the various tests below.

FAQ: Given a series, how do you know which test to use ?
Is there any easy algorithm or recipe to tell which test to use ?
Answer: NO ! Actually you are asking a "wrong" question.
Get rid of this high-school mentality (math = memorizing the recipe for a solution without understanding).

Given a series, what you should do is:
Step 1. Look for a series which is simpler yet similar to the given series.
Step 2. Determine whether that simpler series converges or diverges.
Step 3. The original series accordingly should converge or diverge. Justify your judgement by using an appropriate test.
Warning: There could be many tests to justify your answer in Step 3.

Summary of Tests:

(1) p**-series**

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & \text{when} \quad p > 1 \\ \text{diverges} & \text{when} \quad p \le 1 \end{cases}$$

(2) **Geometric seies**

$$\sum a \cdot r^{n-1} \ (a \neq 0) \begin{cases} \text{converges} & \text{when} \quad |r| < 1 \\ \text{diverges} & \text{when} \quad |r| \ge 1 \end{cases}$$

(3) **Comparison Test**

1st form: $0 \le a_n \le b_n, \sum b_n$ converges $\Longrightarrow \sum a_n$ converges.
2nd form: $0 \le b_n \le a_n, \sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

(4) **Limit Comparison Test**

$$0 < a_n, b_n \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$
$$\Longrightarrow \sum a_n \& \sum b_n \text{ share the same destiny.}$$

Note: When $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, we cannot conclude $\sum a_n$ and $\sum b_n$ share the same destiny in general. One exception is:

$$0 < a_n, b_n \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = 0, \sum b_n \text{ converges}$$
$$\Longrightarrow \sum a_n \text{ also converges.}$$

(5) **Test for Divergence**

$\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n$ diverges.
Warning: There is NO (Test for Convergence), which says
$\lim_{n\to\infty} a_n = 0 \Longrightarrow \sum a_n$ converges.

(6) **Alternating Series Test**

$\sum a_n = \sum (-1)^{n-1 \text{ or } n} b_n$ with $b_n > 0$

converges if one checks the two conditions

$$\begin{cases} \text{(i)} & b_n \geq b_{n+1}, \\ \text{(ii)} & \lim_{n\to\infty} b_n = 0. \end{cases}$$

Note: If condition (ii) fails, i.e., if $\lim_{n\to\infty} b_n \neq 0$, then

$\lim_{n\to\infty} b_n \neq 0 \Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n$ diverges.

Meanwhile, the failure of condition (i) does not necessarily imply that $\sum a_n$ diverges.

(7) **Absolute Convergence**

$\sum |a_n|$ converges $\Longrightarrow \sum a_n$ converges.

When this happens, we say $\sum a_n$ absolutely converges.

Note: When $\sum a_n$ converges even though $\sum |a_n|$ diverges, we say $\sum a_n$ conditionally converges.

(8) **Ratio Test**

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \begin{cases} < 1 & \Longrightarrow & \sum a_n \text{ (abs.) converges} \\ > 1 & \Longrightarrow & \sum a_n \text{ diverges} \\ = 1 & & \text{inconclusive} \end{cases}$$

(9) **Root Test**

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} < 1 & \Longrightarrow & \sum a_n \text{ (abs.) converges} \\ > 1 & \Longrightarrow & \sum a_n \text{ diverges} \\ = 1 & & \text{inconclusive} \end{cases}$$

(10) **Integral Test**

$a_n = f(n)$ where f is a continuous, positive, and decreasing function on $[1, \infty)$, then $\displaystyle\int_1^\infty f(x)dx$ and $\sum a_n$ share the same destiny.

Example Problems

2.1. Judge whether the following series converges or diverges:

①. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

② $\sum_{n=2} \frac{1}{\ln n}$

③ $\sum_{n=3} \frac{1}{n^2-5} \sin(n)$

④ $\sum_{n=1} \frac{n+\sqrt{n}}{n^2}$

⑤ $\sum_{n=2} (-1)^n \ln\left(\frac{1}{n}\right)$

⑥ $\sum_{n=2} \frac{5+\sqrt{n}}{n^2-n}$

⑦ $\sum_{n=1} \frac{n^3}{e^{n^4}}$

⑧ $\sum_{n=2} \left(\frac{1}{\ln(n)} - \frac{1}{\ln(n+1)}\right)$

⑨. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

⑩. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

⑪. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

⑫. $\sum_{n=1}^{\infty} (-1)^n \arctan\left(\frac{1}{n}\right)$

⑬. $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$

⑭. $\sum_{n=1}^{\infty} (-1)^n \arctan(n)$

⑮. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$

⑯. $\sum_{n=1}^{\infty} \frac{e^n+3n}{e^{5n}}$

⑰. $\sum_{n=1}^{\infty} \frac{7+3^n}{2+5^n}$

⑱. $\sum_{n=1}^{\infty} \frac{n^2+3^n}{n+5^n}$

3. You are supposed to be able to determine, when the expression for the general term a_n involves a parameter, for what value(s) of that parameter the series $\sum a_n$ converges (or diverges).

Example Problems

3.1. (o) When $\alpha = 1$, does the series $\displaystyle\sum_{n=2} \frac{1}{n^\alpha \ln n} = \sum_{n=2} \frac{1}{n \ln n}$ converge or diverge ?

(i) When $\alpha < 1$, which one is bigger, $\dfrac{1}{n^\alpha \ln n}$ or $\dfrac{1}{n \ln n}$?

Does the series $\displaystyle\sum_{n=2} \frac{1}{n^\alpha \ln n}$ converge or diverge ?

Which test are you using ?

(ii) When $\alpha > 1$, which one is bigger, $\dfrac{1}{n^\alpha \ln n}$ or $\dfrac{1}{n^\alpha}$ (for $n \geq 3$)?

Does the series $\displaystyle\sum_{n=2} \frac{1}{n^\alpha \ln n}$ converge or diverge ?

Which test are you using ?

3.2. For what value(s) of α does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{5\alpha} + 3n^\alpha}{n^{2\alpha}}$ converge ?

3.3. For what value(s) of p, does the series $\displaystyle\sum_{n=1}^{\infty} n^2 \left\{(3 + n^5)^{2p}\right\}$ converge ?

4. You are suppoed to know when $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, Ratio Test is inconclusive (and hence in order to determine whether the series converges or diverges we have to apply some other methods or tests).

Example Problems

4.1. For which of the following series is the Ratio Test inconclusive ?

I. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

II. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{5^n}$

III. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

IV. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

5. You are supposed to know various logical implications/connection to judge whether the series converges or diverges.

Example Problems

5.1. Determine whether the following statements are true or false.

(i) $\lim_{n\to\infty} n^2 \cdot a_n = 3$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

(iii) The series $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}\alpha^n$ converges for all values α with $0 < \alpha < 3$.

(iv) If $0 < b_n < c_n$ for all n, and $\sum_{n=1}^{\infty} c_n$ converges, then $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ converges conditionally.

(v) $\lim_{n\to\infty} |a_n| \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(vi) If $\lim_{n\to\infty} n^2(a_n)^2 = 5$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(vii) If the power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ converges when $x = 5$, then the series converges when $x = 0$.

6. You are supposed to know how to use the Estimation Theorem for Alternating Series to find the minimum number of terms you should compute to have the error stay within the given value.

Example Problems

6.1. The Alternating Series Test shows that

$$\sum_{n=1}^{\infty}(-1)^n \frac{5}{3n}$$

converges to a value S.

Set

$$S_N = \sum_{n=1}^{N}(-1)^n \frac{5}{3n}.$$

Find the smallest N, using The Estimation Theorem for Alternating Series, such that we can conclude

$$|S - S_N| < \frac{1}{10^3}.$$

6.2. In order to estimate the integral

$$\int_0^{0.1} \frac{1}{1+x^3} dx,$$

we use the alternating series

$$S = \left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{3n+1} x^{3n+1} \right]_0^{0.1}.$$

Set

$$S_N = \left[\sum_{n=0}^{N} (-1)^n \frac{1}{3n+1} x^{3n+1} \right]_0^{0.1}.$$

Find the smallest N, using The Estimation Theorem for Alternating Series, such that we can conclude

$$|S - S_N| < \frac{1}{10^7}.$$

7. You are supposed to know how to find the power series expression for a function, using the basic formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

You are also supposed to know how to find the power series expression for its derivatives and integrations.

Example Problems

7.1. Look at Webassign HW 28.

8. Given a power series, you are supposed to know how to determine the radius of convergence by fist using the Ratio Test, and then to determine the interval of convergence by checking th behavior at the boundary points.

Example Problems

8.1. Look at Webasssign HW 27.

9. Given a function, you are supposed to be able to find its Maclaurin series (power series centered at 0) and Taylor series (power series ceentered at a) by the basic formulas

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \end{aligned}$$

Example Problems

9.1.. Look at Webassign HW 29.

10. You are supposed to be able to evaluate the series, by looking at the Maclaurin series for a function and by plugging in the appropriate value.

Example Problems

10.1. Evaluate the following series

$$
\begin{aligned}
&①.\ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!6^{2n}} \\
&②.\ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} \\
&③.\ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!4^{2n+1}} \\
&④.\ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!2^{2n+1}} \\
&⑤.\ \sum_{n=0}^{\infty} \frac{(\ln 3)^n}{n!} \\
&⑥.\ \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}
\end{aligned}
$$