Study Guide for Final Exam

1. You are supposed to be able to calculate the cross product $\vec{a} \times \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, and understand its geometric meaning. As an application, you should be able to compute the area of a parallelogram (or a triangle) in 3-space.

   Example Problems

   1.1. Find the unit vector which is perpedicular to the following two vectors

$$\begin{cases} \vec{a} & = & \langle 1, 0, 3 \rangle \\ \vec{b} & = & \langle 2, -5, 4 \rangle \end{cases},$$

   and whose z-componet is negative.

   1.2. Find the area of the parallelgram in the 3-space formed by the following 4 points:

$$\begin{cases} P & = & (3, -4, 5) \\ Q & = & (4, -3, 10) \\ R & = & (-1, 2, 7) \\ S & = & (-2, 1, 2) \end{cases}$$

2. You are supposed to be able to calculate the dot product $\vec{a} \cdot \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$, and understand its geometrical meaning. As an application, you should be able to use the dot product for finding the angle between two vectors.

   Example Problems

   2.1. We have two vectors in the 3-space $\vec{a}$ with $|\vec{a}| = 3$ and $\vec{b}$ with $|\vec{b}| = 5$, forming an angle θ between them. Compute

$$(\vec{a} \cdot \vec{b})^2 + (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}).$$

   2.2. Find the angle between the tangent line to the curve $y = x^2$ at $(1, 1)$, and the tangent line to the curve $y = x^3$ at $(1, 1)$.

3. You are supposed to be able to compute the vector projection $\mathrm{proj}_{\vec{a}}\vec{b}$ and scalar projection $\mathrm{comp}_{\vec{a}}\vec{b}$ of a vector $\vec{b}$ onto $\vec{a}$ by the formulas

$$\begin{cases} \mathrm{proj}_{\vec{a}}\vec{b} &= \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\dfrac{\vec{a}}{|\vec{a}|}\right) \\ \mathrm{comp}_{\vec{a}}\vec{b} &= \dfrac{\vec{a}\cdot\vec{b}}{\sqrt{\vec{a}\cdot\vec{a}}} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \end{cases}$$

Example Problems

3.1. Find the vector projection of $\vec{b} = \langle 3, -1, 2\rangle$ onto $\vec{a} = \langle 1, 1, 2\rangle$.

3.2. Given the points

$$\begin{cases} P(1, 3, 5), \\ Q(0, 3, -2), \\ R(-1, -1, -1), \end{cases}$$

set

$$\begin{cases} \vec{a} &= \overrightarrow{PR}, \\ \vec{b} &= \overrightarrow{PQ}. \end{cases}$$

Find the number c such that $\mathrm{Proj}_{\vec{a}}\vec{b} = c\left(\dfrac{\vec{a}}{|\vec{a}|}\right)$.

4. You are supposed to be able to compute the scalar triple product of the vectors $\vec{a}, \vec{b}$, and $\vec{c}$ as the determinant of the 3×3 matrix formed by these vectors, and to understand its geometrical meaning as the volume of the parallelepiped formed by these vectors.

Example Problems

4.1. Find the volume of the parallelepiped determined by the following three vectors

$$\begin{cases} \vec{a} &= \langle 3, 4, 0\rangle, \\ \vec{b} &= \langle -1, 2, 1\rangle, \\ \vec{c} &= \langle 0, 1, 5\rangle. \end{cases}$$

4.2. Find the value t so that the following three vectors are collinear

$$\begin{cases} \vec{a} &= \langle 1, 1, 1\rangle, \\ \vec{b} &= \langle -1, 2, 0\rangle, \\ \vec{c} &= \langle 0, 1, t\rangle. \end{cases}$$

5. You are supposed to be able to calculate the area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ by the formula

$$\int_a^b |f(x) - g(x)| \, dx.$$

Example Problems

5.1. Find the area of the region bounded by the curves $y = |x|$ and $y = 2 - x^2$.

5.2. Find the area of the region bounded by the curves $y = \cos x$ and $y = \sin x$ between $x = 0$ and $x = \pi/2$.

6. You are supposed to be able to calculate the volume of the solid obtained by rotation, using

◦ Washer method, and

◦ Cylindrical Shell method.

Example Problems

6.1. Consider the region bounded by the curves $y = -x^2 + 2x + 3$ and $x + y = 3$.

(i) Compute the volume of the solid obtained by rotating the region about the x-axis by the Washer Method.

(ii) Compute the volume of the solid obtained by rotating the region about the y-axis by the Cylindrical Shell method.

6.2. Find the volume of the solid generated by revolving about the y–axis the region bounded by $y = 0, y = \dfrac{1}{1 + x^4}$, $x = 0$, and $x = 1$.

6.3. Consider the region bounded by the curves $y = -(x - 3)^2 + 9$ $(0 \le x \le 3)$, $y = 0$, and $x = 3$.

We want to compute the volume of the solid obtained by rotating the region about the line $x = 5$. Write down the formula to compute the volume by

(i) the Washer Method, and

(ii) the Cylindrical Shell method.

7. You are supposed to be able to compute the volume of a solid when the descriptions of its base and the perpendicular cross section are given.

Example Problems

7.1. Find the volume of a solid

(i) whose base is region enclosed by an ellipse $\dfrac{x^2}{4} + y^2 = 1$,

(ii) whose cross sections, perpendicular to the base and the x-axis, are isosceles right triangles with (the hypotenuses on the base).

8. You are supposed to be able to determine whether the given improper integral converges or diverges, and when it converges, to be able to evaluate its value.

Example Problems

8.1. Determine whether the following improper integral converges or diverges, and when it converges, evaluate its value.

I. $\displaystyle\int_0^2 \frac{dx}{\sqrt[3]{x}}$

II. $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$

III. $\displaystyle\int_1^\infty \frac{xdx}{1+x^2}$

IV. $\displaystyle\int_1^3 \frac{dx}{x-2}$

9. You are supposed to be able to calculate the average f_{ave} of a function $y = f(x)$ over the interval $[a, b]$ by the formula

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

Example Problems

9.1. Compute the average of the function $f(x) = \tan^3 x$ over the interval $[0, \pi/3]$.

9.2. Compute the average of the function $f(x) = \ln x$ over the interval $[1, 3]$.

10. You are supposed to be able to calculate the work required to empty the tank, to stretch the spring, and to lift the chain, etc..

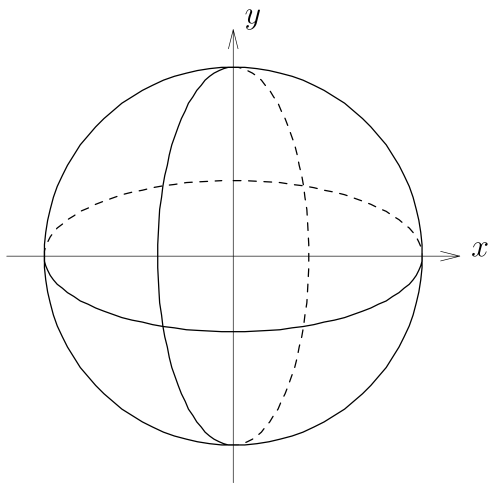
Example Problems

10.1. Suppose a force of 6 lbs is required to stretch a spring 2 feet beyond its natural length. How much work is required to stretch the spring an additional foot ?

10.2. A 12 ft chain weighs 15 lbs and hangs from a ceiling. Compute the work needed to lift the lower end so that it is level with the upper end.

10.3. A spherical tank 6 feet in diameter is full of water (density: 62.5 lb/ft^3).

Write down the formula for the work required to pump all the water in the tank to the level 5 feet above the top of the tank (let the origin be at the center of the tank and the y–axis point upward).



11. You are supposed to be able to calculate the arclength L of a curve $y = f(x), a \le x \le b$ by the formula

$$L = \int_a^b \sqrt{1 + \{f'(x)\}^2}\, dx.$$

When the curve is given by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta$, then the formula becomes

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{\{f'(x)\}^2 + \{g'(t)\}^2}\, dt.$$

Example Problems

11.1. Compute the length of the curve

$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x) \quad (1 \le x \le 2).$$

11.2. Compute the arclength of the curve given parametrically by

$$x = \sin^2 t, \quad y = \frac{\cos 2t}{2}, \quad 0 \le t \le \frac{\pi}{2}.$$

12. You are supposed to be able to calculate the Maclaurin series and Taylor series (centered at $x = a$) of a given function by the formulas

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \end{aligned}$$

and understand the relation between the coefficients of these series and the (higher) derivatives.

Example Problems

12.1. Find the Maclaurin series for $f(x) = \dfrac{1}{x^2 - x - 6}$.

12.2. Find the Maclaurin series for $f(x) = \tan^{-1}(3x)$.

12.3. Find the coefficient of $(x-2)^7$ in the Taylor series for $\ln(3-x)$ centered at $a = 2$..

12.4. Find the Maclaurin series for $f(x) = x^3 \sin^2(x)$.

12.5. Given a Maclaurin series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{5^n}{(n+4)!}(x-3)^n$, find the 8th derivative $f^{(8)}(3)$ of f at $x = 3$.

13. You are supposed to be able to compute the integration of the form

$$\begin{aligned} &\int \sin^m(x) \cos^n(x)\ dx \\ &\int \tan^m(x) \sec^n(x)\ dx. \end{aligned}$$

Example Problems

13.1. Compute the following integrals.

(i) $\displaystyle\int \sin^2 x\ dx$

(ii) $\displaystyle\int_{\pi/3}^{\pi/2} \sin^3 x\ dx$

(iii) $\displaystyle\int \tan^3 x \sec x\ dx$

(iv) $\displaystyle\int \sec^4 x \tan^3 x\ dx$

(v) $\displaystyle\int \tan^3 x \cos x\ dx$

14. You are supposed to be able to evaluate the integral using Integration by Parts

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

or equivalently

$$\int udv = uv - \int vdu.$$

Example Problems

14.1. Compute the following integrals.

(i) $\displaystyle\int x^2 e^x\ dx$

(ii) $\displaystyle\int x\cos x\ dx$

(iii) $\displaystyle\int e^x \sin x\ dx$

15. You are supposed to be able to compute the integration of the form

$$\int \frac{P(x)}{Q(x)}\ dx$$

with $P(x), Q(x)$ polynomials, using Partial Fractions.

Example Problems

15.1. Determine the proper form of partial fractions for

$$\frac{x^5 + x}{(x-2)(x^2-4x-5)(x^2-4x+5)}.$$

15.2. Compute the following integrals.

(i) $\displaystyle\int \frac{1}{x^2-5x-14}\ dx$

(ii) $\displaystyle\int \frac{1}{x^3-1}\ dx$

(iii) $\displaystyle\int_3^{3+1/\sqrt{3}} \frac{1}{x^2-6x+10}\ dx$

16. You are supposed to be able to evaluate the integral using Trigonometric Substitution

$$\begin{cases} \sqrt{a^2 - x^2}, & x = a\sin\theta, & dx = a\cos\theta d\theta, & \sqrt{a^2 - x^2} = a\cos\theta \\ \sqrt{x^2 + a^2}, & x = a\tan\theta, & dx = a\sec^2\theta d\theta, & \sqrt{x^2 + a^2} = a\sec\theta \\ \sqrt{x^2 - a^2}, & x = a\sec\theta, & dx = a\sec\theta\tan\theta d\theta, & \sqrt{x^2 - a^2} = a\tan\theta \end{cases}$$

Example Problems

16.1. Evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{2 - x^2}}\, dx.$$

16.2. Find a formula for the following indefinite integral

$$\int \frac{dx}{x^2\sqrt{1 - x^2}}.$$

16.3. Transform the following integral into a trigonometric integration

$$\int \frac{\sqrt{7x^2 - 1}}{x^2}\, dx$$

using an appropriate trigonometric substitution.

16.4. Evaluate the following integral

$$\int_4^{11/2} \frac{1}{\sqrt{-x^2 + 8x - 7}}\, dx.$$

17. You are supposed to be able to judge whether the given series (absolutely, conditionally) converges or diverges by using various tests.

Example Problems

17.1. Which of the following series converges conditionally?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ III. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)^n$

IV. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n^2 + 5n}}$ V. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n}$ VI. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$

17.2. Judge whether the following series converges or diverges.

(i) $\sum_{n=1}^{\infty} \sin(n) e^{1/n}$

(ii) $\sum_{n=1}^{\infty} \sin(1/n) e^{1/n}$

(iii) $\sum_{n=1}^{\infty} \sin(1/n^2) e^{1/n}$

(iv) $\sum_{n=1} \ln\left(1 + \left(\frac{1}{n}\right)^2\right)$

(v) $\sum_{n=1} \{\sqrt{n^3} - \sqrt{n^3 - 1}\}$

17.3. Find the range of p (resp. q) such that the following series (a) (resp. (b)) converges.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n+1}\right)^p$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^q}{2}$

18. You are supposed to be able to give an estimate of the alternating series within a given error using the Estimation Theorem for the Alternating Series. Sometimes this method is used to give an estimate of the integral in terms of the power series.

Example Problems

18.1. Use power series to approximate $\displaystyle\int_0^{0.1}\frac{x^3}{1+x^2}\,dx$ with error smaller than 10^{-7}.

18.2. Using Maclaurin series and the Estimation Theorem for alternating series, we can obtain the approximation

$$\int_0^{0.1} x\sin x\,dx \approx \frac{1}{3}(0.1)^3 - \frac{1}{30}(0.1)^5 \text{ with error } \leq c.$$

Find the value c.

19. You are supposed to be able to calculate the area of the surface obtained by rotating the curve about the x-axis by the formula

$$\begin{aligned} A &= \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_c^d 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy. \end{aligned}$$

When the curve is rotated around the y-axis the formula becomes

$$\begin{aligned} A &= \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_c^d 2\pi x\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy. \end{aligned}$$

Example Problems

19.1. The curve $y=\sqrt{4-x^2}$, $0\leq x\leq 1$, is an arc of the circle $x^2+y^2=4$.

(i) Find the area of the surface obtained by rotating this arc about the x-axis.

(ii) Find the area of the surface obtained by rotating this arc about the y-axis.

19.2. We would like to compute the area of the surface generated by rotating the curve $y = 3\sin x (0 \le x \le \pi/2)$ about the x-axis. In order to compute the area, write down

I. the formula which is the integration with respect to dx, and

II. the formula which is the integration with respect to dy.

20. You are supposed to be able to determine the radius of convergence and the interval of convergence for the given power series.

Example Problems

20.1. Determine the interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{\sqrt{n+2}}(x-7)^n$.

20.2. Find the radius of convergence for $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n^n}{(n+2)!}(x-1)^n$.

20.3. Determine the interval of convergence for the Taylor series

$$\sum_{n=2}^{\infty} \frac{3^n}{\ln n}(x-4)^n.$$

21. You are supposed to be able to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve defined by the parametric equations

$$\begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

by the formulas

$$\begin{cases} \dfrac{dy}{dx} &= \dfrac{dy/dt}{dx/dt} &= \dfrac{g'(t)}{f'(t)} \\ \dfrac{d^2y}{dx^2} &= \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} &= \dfrac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)}. \end{cases}$$

Example Problems

21.1. Find the slope $\frac{dy}{dx}$ of the line tangent to the curve given parametrically by

$$\begin{cases} x &= \tan t, \\ y &= \sec t, \end{cases}$$

and the value of the second derivative $\frac{d^2y}{dx^2}$ at $t = \pi/4$.

21.2. You are given the parametric equation for the cycloid

$$\begin{cases} x &= r(\theta - \sin\theta) \\ y &= r(1 - \cos\theta) \end{cases}$$

Find the formulas for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of θ.

22. You are supposed to know how to add, subtract, multiply, and divide complex numbers. Especially the division is carried out by multiplying the complex conjugate of the denominator, i.e.,

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

Example Problems

22.1. Compute the following.

(i) $(5-3i)+(7+4i)$

(ii) $(5-3i)(7+4i)$

(iii) $\dfrac{5-3i}{7-4i}$

23. You are supposed to know how to express a complex number in polar coordinates, determining its modulus and argument, and how the multiplication and division of two complex numbers are carried out in polar coordinates.

Example Problems

23.1.

(i) Find the expressions of the following complex numbers in polar coordinates, determining their moduli and arguments.

$$\begin{cases} z &= 3\sqrt{2}+3\sqrt{2}i \\ w &= -4+4\sqrt{3}i \end{cases}$$

(ii) Find the expressions of $zw, z/w, z^2/w^3$ in polar coordinates.

24. You are supposed to know how to use De Moivre's Theorem to compute the powers of complex numbers, and solve the equation of the form $z^n = a+bi$.

Example Problems

24.1. Compute

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{121}.$$

24.2. Find all the solutions for the equation $z^6 = 64$.

24.3. Solve the equation $z^5 = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.

25. You are supposed to be able to draw the picture of a curve defined by the equation given in polar coordinates.

Example Problems

25.1. Sketch the following curves given by the polar equations.

(i) $r = 2\cos\theta$

(ii) $r = 3\cos 2\theta$

(iii) $r = 2\cos 3\theta$

26. You are supposed to transform the equation in polar coordinates into the one in Cartesian coordinates.

Example Problems

26.1. Find the equation in terms of the Cartesian coordinates for the curve given by the polar equation $r = \tan\theta\sec\theta$.

27. You are supposed to be able to compute the coordinates of the centroid (the center of mass) of the region enclosed by the curves in the xy-plane of uniform density.

Example Problems

27.1. Find the centroid of the region enclosed by the curves $y = 4$ and $y = x^2$ in the xy-plane of uniform density.

27.2. Find the centroid of the triangle enclosed by the lines $y = x - 5$, $2x + y = 3$, and $-3x + y = 3$ in the xy-plane of uniform density.

28. You are supposed to be able to compute some of the non-trivial and difficult limits, which often show up carrying out the Ratio Test and Root Test.

Example Problems

(i) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

(ii) $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} / \frac{n!}{n^n}$

(iii) $\lim_{n\to\infty} n^{1/n}$

(iv) $\lim_{n\to\infty} \frac{n!}{1 \cdot 3 \cdot \cdots \cdot (2n-1)}$