Answer Keys for Study Guide for Exam 3

1. 1.

Step 1.

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on $[-1, 4]$

Step 2.

1. $a = -1$, $b = 4$

2. $$f'(x) = 12x^3 - 48x^2 + 36x$$
$$= 12x(x^2 - 4x + 3)$$
$$= 12x(x-1)(x-3)$$

$f'(c) = 0$, $c = 0, 1, 3$

Step 3.

$f(-1) = 37$ ← abs. max.

$f(4) = 32$

$f(0) = 0$

$f(1) = 3$

$f(3) = -27$ ← abs. min.

1.2

(2)

(a) Step 1

$f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

Step 2

(1) $a = -2$, $b = 3$

(2) $f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x+1)(x-2)$

$f'(c) = 0$, $c = -1, 2$

Step 3.

$f(-2) = -3$

$f(3) = -8$

$f(-1) = 8$ ← abs. max.

$f(2) = -19$ ← abs. min.

(b) Step 1. 3

$$f(x) = x e^{\frac{x}{2}} \quad \text{on} \quad [-3, 1]$$

Step 2.

① $a = -3$, $b = 1$

② $f'(x) = 1 \cdot e^{\frac{x}{2}} + x \cdot \frac{1}{2} e^{\frac{x}{2}}$

$$= \left(1 + \frac{x}{2}\right) e^{\frac{x}{2}}$$

$f'(c) = 0 \qquad c = -2$

Step 3.

$f(-3) = -3 e^{-\frac{3}{2}}$

$f(1) = 1 \cdot e^{\frac{1}{2}}$ ← abs. max

$f(-2) = -2 e^{-\frac{2}{2}}$ ← abs. min.

Note :

x	-3		-2		1
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$	

$\longrightarrow \quad f(-3) > f(-2)$

(c) Step 1.

$f(x) = (x^2-1)^3$ on $[-1, 3]$

Step 2

① $a = -1$, $b = 3$

② $f'(x) = 3(x^2-1)^2 \cdot 2x$

$f'(c) = 0$, $c = \pm 1$, 0

Step 3.

$f(-1) = 0$

$f(3) = 512$ ← abs. max

$f(1) = 0$

$f(0) = -1$ ← abs. min

④

(d) Step 1

$f(t) = 2\cos t + \sin 2t$ on $[0, 2\pi]$

Step 2

① $a = 0$, $b = 2\pi$

② $f'(t) = -2\sin t + 2\cos 2t$

$= -2\sin t + 2(1 - 2\sin^2 t)$

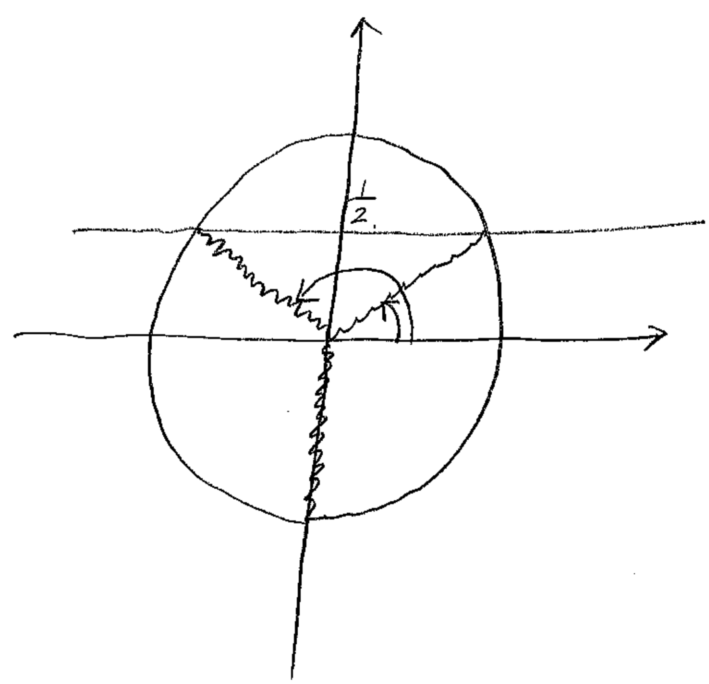
$= -2(2\sin^2 t + \sin t - 1)$

$= -2(2\sin t - 1)(\sin t + 1)$

$f'(c) = 0$

$c = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{3\pi}{2}$



Step 3.

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$f(0) = 2.$

$f(2\pi) = 2.$

$f\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$ ← abs. max

$f\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$ ← abs. min

$f\left(\frac{3\pi}{2}\right) = 0$

(e) Step 1.

$$f(x) = \ln(x^2 + x + 1) \text{ on } [-1, 1]$$

$$\left(\text{Note: } \begin{aligned} & x^2 + x + 1 \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0. \end{aligned}\right)$$

Step 2

① $a = -1, \quad b = 1$

② $$f'(x) = \frac{2x + 1}{x^2 + x + 1}$$

$f'(c) = 0 \qquad c = -\frac{1}{2}$

Step 3.

$f(-1) = 0$

$f(1) = \ln 3 > 0 \leftarrow$ abs. max

$f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right) < 0 \leftarrow$ abs. min

(f) Step 1.

$$f(x) = x - \sqrt{x} \quad \text{on} \quad [0, 9]$$

Step 2.

① $a = 0$, $b = 9$

②

$$f'(x) = 1 - \frac{1}{2\sqrt{x}}$$

for $x \in (0, 9)$

$f'(c) = 0 \qquad c \in (0, 9)$

$$c = \frac{1}{4}$$

Step 3.

$f(0) = 0$

$f(9) = 6 \quad \leftarrow$ abs. max

$f\left(\frac{1}{4}\right) = -\frac{1}{4} \quad \leftarrow$ abs. min

2.1.

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5)$$

x		-2		-1		1		3		5	
$f'(x)$	$-$	0	$-$	0	$+$	0	$-$	0	$-$	0	$+$
$f(x)$	$\searrow$		$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

(a) local max. $x = 1$.

(b) local min $x = -1, 5$.

2.2. (10)

(a) $f(x) = x^8(x-4)^7$.

$f'(x) = 8x^7(x-4)^7$
$+ x^8 \cdot 7(x-4)^6$

$= x^7(x-4)^6 \{8(x-4) + x \cdot 7\}$

$= x^7(x-4)^6(15x-32)$

(b) $f''(x)$

$= 7x^6 \cdot (x-4)^6(15x-32)$
$+ x^7\, 6(x-4)^5(15x-32)$
$+ x^7 \cdot (x-4)^6 \cdot 15$

$= x^6(x-4)^5 \cdot \{7(x-4)(15x-32)$
$+ x\,(15x-32)$
$+ x(x-4) \cdot 15\}$.

(b)

$x = 0 \qquad f''(0) = 0.$

$\rightarrow$ 2nd Der. Test inconclusive

$x = \frac{32}{15}. \qquad f''\left(\frac{32}{15}\right)$

$= \left(\frac{32}{15}\right)^6 \left(\frac{32}{15} - 4\right)^5 \left\{\frac{32}{15}\left(\frac{32}{15} - 4\right) \cdot 15\right\}$

$\quad\ \vee \qquad\quad \wedge \qquad\qquad\quad \wedge$

$\quad\ 0 \qquad\quad 0 \qquad\qquad\quad 0.$

> 0

$\rightarrow$ 2n Der. Test

local min.

$x = 4. \qquad f''(4) = 0.$

$\rightarrow$ 2nd Der. Test inconclusive

(c)

x		0		$\frac{32}{15}$		4	
$f'(x)$	$+$	0	$-$	0	$+$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$

$x = 0$ local max

$x = \frac{32}{15}$ local min

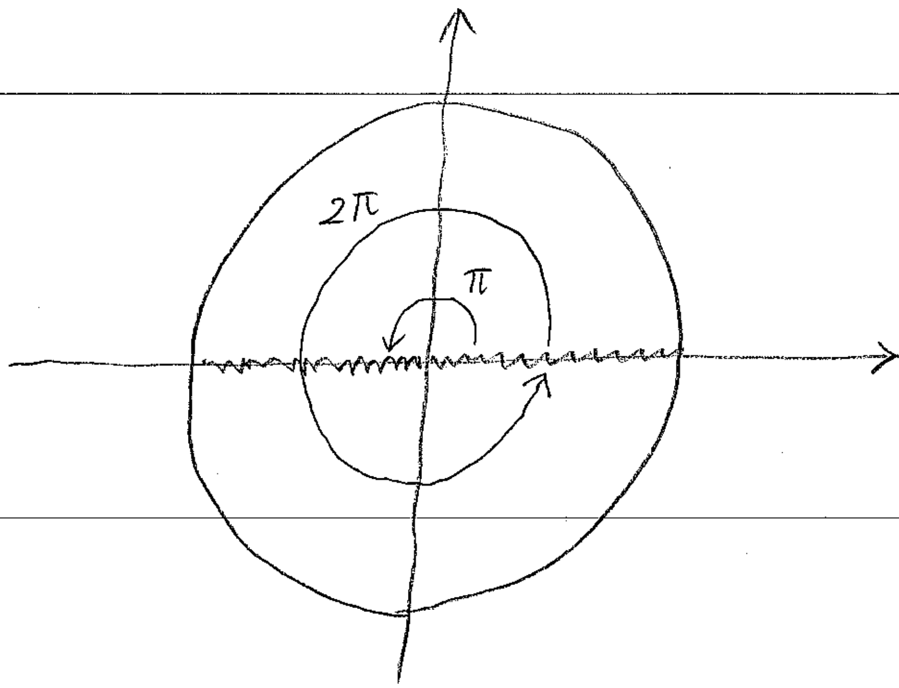
$x = 4$ neither local max nor local min

3.1.

$$f(x) = \frac{1}{2}x - \sin x \quad \text{on } (0, 3\pi)$$

$$f'(x) = \frac{1}{2} - \cos x$$

$$f''(x) = \sin x$$



x	0		π		2π		3π
$f''(x)$	X	+	0	−	0	+	X
	X	concave up		concave down		concave up	X

inf. inf.

3.2

$$f''(x) = (x+5)^3(x+2)^2(x-2)^5(x-3)^3(x-6)^2$$

x		-5		-2		2		3		6	
$f''(x)$	$-$	0	$+$	0	$+$	0	$-$	0	$+$	0	$+$
		infl. pt.				infl. pt.		infl. pt.			

$x = -5, 2, 3.$

These are NOT inflection points since concavity does not change.

3.3

(15)

$$f(x) = x^5 - 5x^4 + 25x$$

$$f'(x) = 5x^4 - 20x^3 + 25$$

$$f''(x) = 20x^3 - 60x^2$$

$$= 20x^2(x-3)$$

x		0		3	
$f''(x)$	$-$	0	$-$	0	$+$
		NOT inf. pt.		inf. pt.	

Ans. Only 1 inflection pt.

4.1.

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(a) $$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \quad \left(\frac{\infty}{\infty}\right).$$

$$= \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

(b) $$\lim_{x\to 0} \frac{1-\cos x}{3x^2} \quad \left(\frac{0}{0}\right).$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \quad \left(\frac{0}{0}\right).$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

(c) $$\lim_{x\to 0} \frac{e^{7x} - \cos 2x}{\tan(3x)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{7e^{7x} + 2\sin 2x}{3\sec^2(3x)} = \frac{7}{3}.$$

(d) $\lim_{x \to 0} \frac{\sin x}{1 - x^2} = 0$

Warning: Do NOT use L'Hospital's rule!

(e) $\lim_{x \to 0} \frac{3x - \sin(3x)}{3x - \tan(3x)}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{3 - 3\cos(3x)}{3 - 3\sec^2(3x)}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\cancel{3} \cdot \cancel{3} \sin(3x)}{-\cancel{3} \cdot 2\sec(3x)\sec(3x)\tan(3x) \cdot \cancel{3}}$

$= \lim_{x \to 0} \frac{\cancel{\sin(3x)}}{-2\sec^2(3x)\frac{\cancel{\sin(3x)}}{\cos(3x)}} = -\frac{1}{2}$

(f) $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{-\sin x}{6x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

5.1.

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(a) $$\lim_{x\to 0} \underbrace{\sin x}_{\downarrow \atop 0} \cdot \underbrace{\ln(2x)}_{\downarrow \atop -\infty} \qquad (0\times(-\infty))$$

$$= \lim_{x\to 0} \frac{\ln(2x)}{1/\sin x}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{2}}{\cancel{2}x}}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0} -\frac{\sin^2 x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} -\frac{2\sin x\cos x}{1} = 0.$$

(b) $$\lim_{x\to\infty} \underbrace{2x}_{\infty} \cdot \underbrace{\tan\left(\frac{1}{3x}\right)}_{0} \qquad (\infty\times 0)$$

$$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{3x}\right)}{\frac{1}{2x}}$$

$$= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{3x}\right)\frac{1}{3}\left(-\cancel{\frac{1}{x^2}}\right)}{\frac{1}{2}\left(-\cancel{\frac{1}{x^2}}\right)} = \frac{2}{3}.$$

(c) $$\lim_{x\to(\frac{\pi}{2})^-} \underbrace{(2x-\pi)}_{\downarrow \atop 0^-} \cdot \underbrace{\tan x}_{\downarrow \atop +\infty}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\tan x}{1/(2x-\pi)}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec^2 x}{-\frac{2}{(2x-\pi)^2}}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} -\frac{(2x-\pi)^2}{2\cos^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to(\frac{\pi}{2})^-} + \frac{\cancel{2}(2x-\pi)\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cos x(+\sin x)}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{2x-\pi}{\cos x \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{2}{-\sin x\sin x + \cos x\cos x} = -2$$

(d) $$\lim_{x \to 1} \left(\underbrace{\frac{x}{x-1}}_{\text{DNE}} - \underbrace{\frac{1}{\ln x}}_{\text{DNE}} \right)$$

$$\left(\lim_{x \to 1} \qquad \text{DNE} \qquad \text{DNE} \right)$$

$$= \lim_{x \to 1} \frac{x \ln x - (x-1)}{(x-1)\ln x}$$

$$= \lim_{x \to 1} \frac{1 \cdot \ln x + x \cdot \frac{1}{x} - 1}{1 \cdot \ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x \to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{1/x}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$$

(e) $$\lim_{x \to 4} \left(\underbrace{\frac{1}{\sqrt{x}-2}} - \underbrace{\frac{4}{x-4}} \right)$$ 21

$$\left(\lim_{x \to 4} \qquad \text{DNE} \qquad \text{DNE} \qquad \right)$$

$$= \lim_{x \to 4} \frac{(\sqrt{x}+2) - 4}{x-4}$$

$$= \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$$

$$= \lim_{x \to 4} \frac{\cancel{\sqrt{x}-2}}{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)} = \frac{1}{4}.$$

6.1.

(a) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x}$ $\quad (1^{\infty})$

Set $y=\left(1+\frac{3}{x}\right)^{7x}$

$$\ln y = 7x\cdot\ln\left(1+\frac{3}{x}\right)$$

Compute

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\underbrace{7x}_{\downarrow\ \infty}\ \times\ \underbrace{\ln\left(1+\frac{3}{x}\right)}_{\downarrow\ 0}$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{1/7x}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot 3\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\frac{1}{7}\cancel{\left(-\frac{1}{x^2}\right)}}=21.$$

Therefore, we conclude

$$\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\ln y \to 21}=e^{21}.$$

(b) $\lim_{x \to \infty} \left(\frac{x+3}{x-2} \right)^{4x+1}$ (1^∞) 23

Set $y = \left(\frac{x+3}{x-2} \right)^{4x+1}$

$$\ln y = (4x+1) \ln \left(\frac{x+3}{x-2} \right)$$

Compute

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \underbrace{(4x+1)}_{\downarrow \atop \infty} \underbrace{\ln \left(\frac{x+3}{x-2} \right)}_{\downarrow \atop 0}$$

$$= \lim_{x \to \infty} \frac{\ln \left(\frac{x+3}{x-2} \right)}{1/(4x+1)} \quad \left(\frac{0}{0} \right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x+3} - \frac{1}{x-2}}{- \frac{4}{(4x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{(x-2) - (x+3)}{(x+3)(x-2)}}{-4/(4x+1)^2}$$

$$= \lim_{x \to \infty} \frac{5 \; (4x+1)^2}{4 \; (x+3)(x-2)} = 20$$

Therefore, we conclude

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\overset{\to 20}{\ln y}} = e^{20}$$

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(c) $\lim_{x \to \infty} \left(2x + e^{5x} \right)^{\frac{1}{x}}$ $\quad (\infty^0)$

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Set $y = \left(2x + e^{5x} \right)^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln \left(2x + e^{5x} \right)$$

Compute

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \underbrace{\frac{1}{x}}_{\downarrow \, 0} \underbrace{\ln \left(2x + e^{5x} \right)}_{\downarrow \, \infty}$$

$$= \lim_{x \to \infty} \frac{\ln \left(2x + e^{5x} \right)}{x} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{2x + e^{5x}} \left(2 + 5e^{5x} \right)}{1}$$

$$= \lim_{x \to \infty} \frac{2 + 5e^{5x}}{2x + e^{5x}} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \to \infty} \frac{5 \cdot 5e^{5x}}{2 + 5 \cdot e^{5x}} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \to \infty} \frac{5 \cdot \cancel{5} \cdot \cancel{5} \cancel{e^{5x}}}{\cancel{5} \cdot \cancel{5} \cdot \cancel{e^{5x}}} = 5.$$

i.e.

$$\lim_{x\to\infty} \ln y = 5$$

Therefore, we conclude

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\overset{\to 5}{\ln y}} = e^{5}.$$

(d) $\lim_{x\to 0^+} \tan(5x)^{\sin x}$ (0^0)

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Set $y = \tan(5x)^{\sin x}$

$\ln y = \sin x \ln\{\tan(5x)\}$

Compute

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \underbrace{\sin x}_{\to 0^+} \ \underbrace{\ln\{\overbrace{\tan(5x)}^{\to 0^+}\}}_{\to -\infty}$$

$$= \lim_{x\to 0^+} \frac{\ln\{\tan(5x)\}}{1/\sin x} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\tan(5x)} \cdot \sec^2(5x) \cdot 5}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{\cos(5x)}{\sin(5x)} \cdot \sin^2 x \cdot 5}{-\cos x \cdot \cos^2(5x)}$$

$$= \lim_{x\to 0^+} \underbrace{\frac{\cos(5x)\cdot 5}{-\cos x \cdot \cos^2(5x)}}_{\to 5} \cdot \underbrace{\frac{\sin^2 x}{\sin(5x)}}_{\to 0} = 0$$

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Note:

$$\left(\lim_{x\to 0^+} \frac{\sin^2 x}{\sin 5x} \quad \left(\frac{0}{0}\right) = \lim_{x\to 0^+} \frac{2\sin x\cos x}{5\cos 5x} = 0. \right)$$

i.e.

$$\lim_{x\to 0^+} \ln y = 0.$$

Therefore, we conclude

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\overset{\to 0}{\ln y}} = e^0 = 1.$$

7.1. (29)

$$f(x) = x^3 - x$$

cont. over $[0, 2]$

diff. over $(0, 2)$

$\rightarrow$ satisfies the conditions of M. V. T.

$\rightarrow$ $\exists\ c \in (0, 2)$

s.t. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$$f'(c) = 3c^2 - 1$$

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2 - 0} = 3$$

$$3c^2 - 1 = 3$$

$3c^2 = 4$ $\quad c^2 = \dfrac{4}{3}$ $\quad c = \pm\sqrt{\dfrac{4}{3}}$

$$c = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3} \in (0, 2)$$

7.2.

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$$f(x) = x^{\frac{2}{3}}$$

cont. over $[-1, 1]$

but NOT diff. over $(-1, 1)$

($f'(x)$ DNE when $x = 0$)

→ does NOT satisfy the conditions for M. V. Th.

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - 1}{1-(-1)} = 0$$

Since

$f'(c) = \frac{2}{3} c^{-\frac{1}{3}}$ when $c \in (-1, 1)$ and $c \neq 0$.

$f'(c)$ DNE when $c = 0$,

there is NO value $c \in (-1, 1)$

s.t. $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = 0$.

7.3

$$f(x) = x^{\frac{1}{5}}$$

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cont. over $[-1, 1]$

but NOT diff. over $(-1, 1)$

($f'(x)$ DNE at $x = 0$)

$\rightarrow$ does NOT satisfy the conditions for M. V. Th

$$\frac{f(1) - f(-1)}{1-(-1)} = \frac{1-(-1)}{1-(-1)} = 1.$$

$$f'(c) = \frac{1}{5} c^{-\frac{4}{5}} \text{ when } c \neq 0.$$

($f'(c)$ does NOT exist when $c = 0$)

$$\frac{1}{5} c^{-\frac{4}{5}} = 1 \rightarrow c^{-\frac{4}{5}} = 5$$

$$\rightarrow c = 5^{-\frac{5}{4}}$$

When $c = 5^{-\frac{5}{4}}$, $f'(c) = \dfrac{f(1) - f(-1)}{1-(-1)}$

(even though we can NOT apply M. V. Th.)

7.4.

(32)

Consider the function

$$f(x) = \sin^{-1} x + \cos^{-1} x$$

over $[-1, 1]$

Then

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right) = 0$$

over $(-1, 1)$

$\longrightarrow$

$f(x)$ constant over $[-1, 1]$

$\longrightarrow$

$$f\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)$$

$$\parallel$$

$$f(0) = \sin^{-1}(0) + \cos^{-1}(0)$$

$$= 0 + \frac{\pi}{2}$$

$$= \frac{\pi}{2}.$$

7.5.

Consider the function

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$$f(x) = \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}(x)$$

over $(0, \infty)$

$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{1+x^2}$$

$$= \frac{1}{x^2+1}(-1) + \frac{1}{1+x^2} = 0$$

$\longrightarrow$ $f(x)$ constant over $(0, \infty)$

$\longrightarrow$

$$f(7) = \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(7)$$

$\shortparallel$

$$f(1) = \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}(1)$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

7.6.

Consider the function

$$f(x) = x^3 + x - 1.$$

f cont. on $[0, 1]$

$$f(0) = -1 < \underset{N.}{\overset{}{0}} < 1 = f(1)$$

$\xrightarrow{\text{I.M.V.Th.}}$ $\exists\, c \in (0, 1)$ s.t. $f(c) = N = 0.$

That is to say, there exists at least one solution c on the interval $[0, 1]$

We claim that this c is the unique solution.

Suppose there is another solution $c' \in [0, 1]$ $(c' \neq c)$ s.t. $f(c') = 0$.

(Say $c < c'$.)

Then by M.V.Th. we should have

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$$d \in (c, c')$$

s.t.

$$f'(d) = \frac{f(c') - f(c)}{c' - c} = \frac{0 - 0}{c' - c} = 0.$$

On the other hand,

$$f'(d) = 3d^2 + 1 > 0.$$

This is a contradiction!

That means there is NO solution c' on $[0, 1]$ other than c.

Conclusion:

# of solution(s) on $[0, 1]$

$= \underline{1}$.

8.1.

(a) $y = f(x) = \dfrac{x}{x^2-16}$

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Domain $x^2 - 16 \neq 0$

$\iff x \neq \pm 4$

$\iff$

$(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$$f'(x) = \frac{1 \cdot (x^2-16) - x \cdot 2x}{(x^2-16)^2}$$

$$= \frac{-(x^2+16)}{(x^2-16)^2}$$

$$f''(x) = \frac{-2x(x^2-16)^2 + (x^2+16)2(x^2-16) \cdot 2x}{(x^2-16)^4}$$

$$= \frac{-2x(x^2-16) + (x^2+16)4x}{(x^2-16)^3}$$

$$= \frac{2x(x^2+48)}{(x^2-16)^3}$$

x		-4		0		4	
$f'(x)$	$-$	X	$-$	$-$	$-$	X	$-$
$f''(x)$	$-$	X	$+$	0	$-$	X	$+$
$f(x)$	↘	X	↘		↘	X	↘

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{x}{x^2-16} = -\infty$$

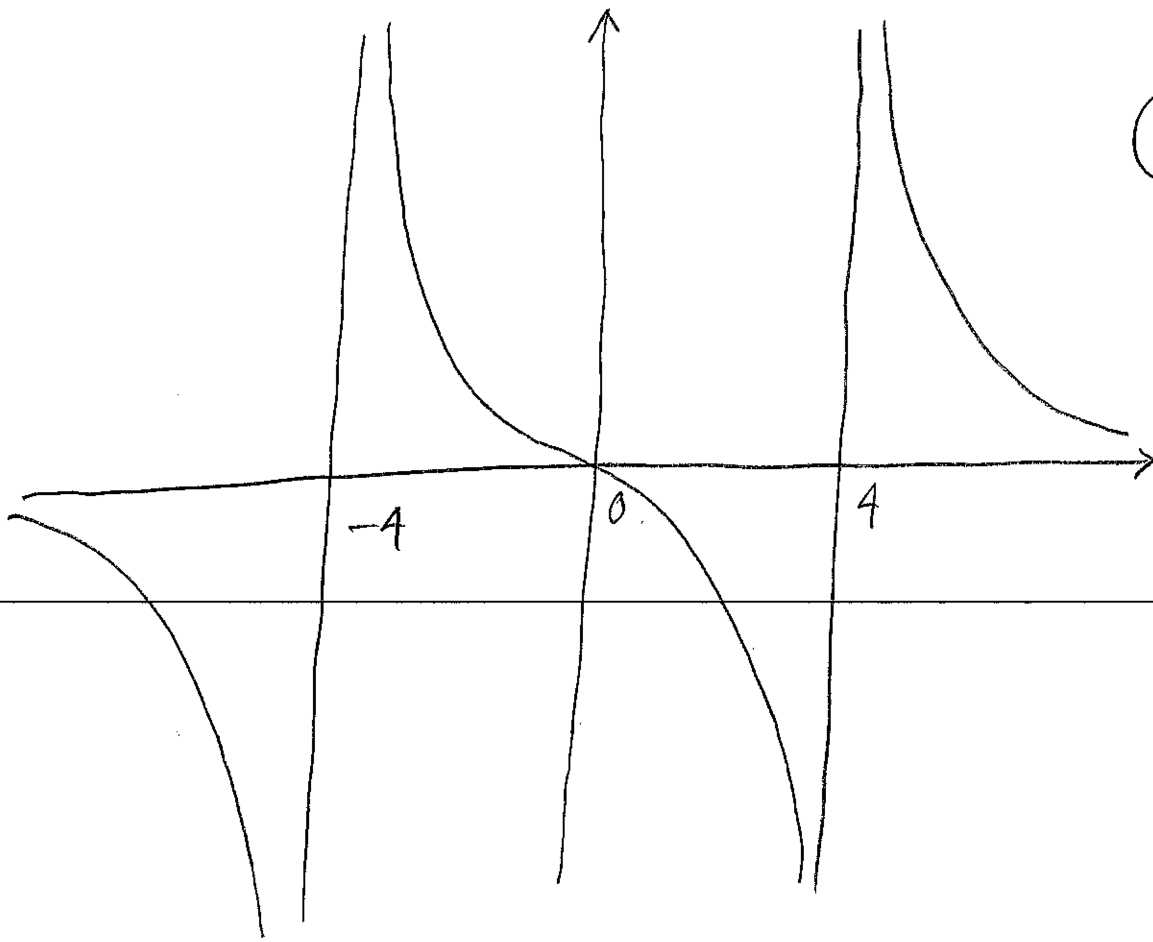
$$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \frac{x}{x^2-16} = +\infty$$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{x}{x^2-16} = -\infty$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{x}{x^2-16} = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{x^2-16} = 0$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2-16} = 0.$$



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(b) $y = f(x) = \dfrac{x}{x^2+16}$

39

Domain $(-\infty, \infty)$

$$f'(x) = \frac{1\cdot(x^2+16) - x\cdot 2x}{(x^2+16)^2}$$

$$= \frac{-(x^2-16)}{(x^2+16)^2}$$

$$= \frac{-(x+4)(x-4)}{(x^2+16)^2}$$

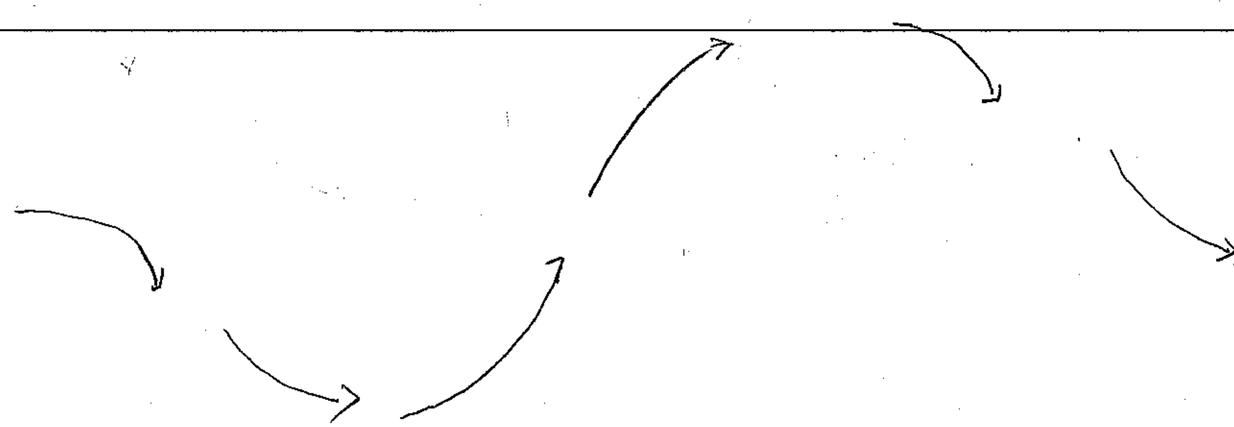
$$f''(x) = \frac{(-2x)(x^2+16)^2 + (x^2-16)\,2(x^2+16)\cdot 2x}{(x^2+16)^4}$$

$$= \frac{(-2x)(x^2+16) + (x^2-16)\cdot 4x}{(x^2+16)^3}$$

$$= \frac{2x^3 - 96x}{(x^2+16)^3}$$

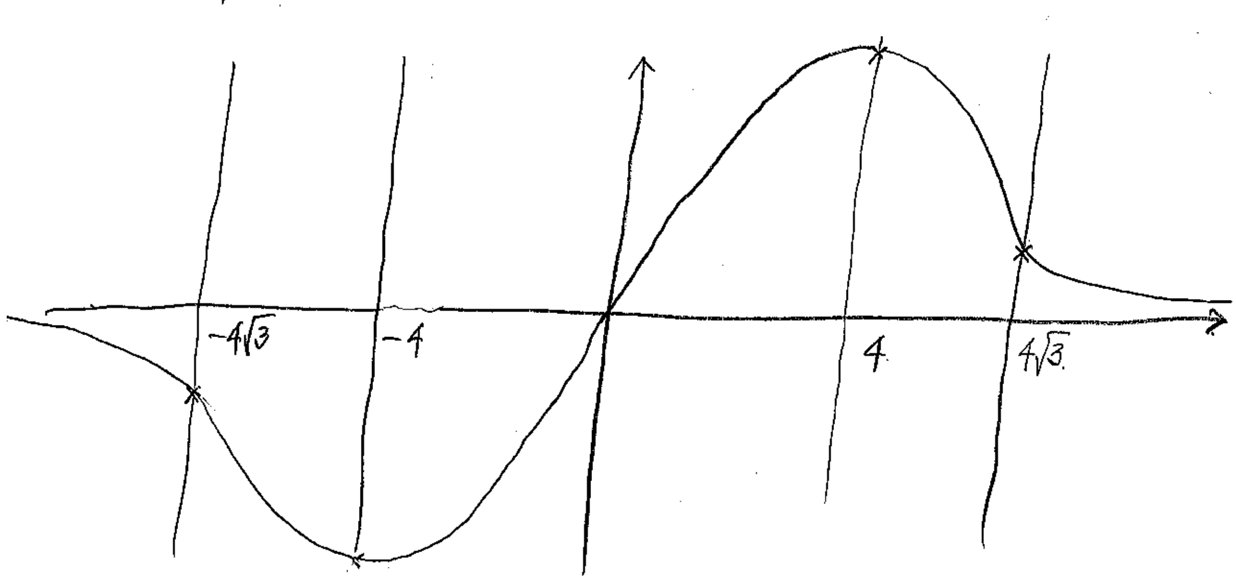
$$= \frac{2x\,(x+4\sqrt{3})(x-4\sqrt{3})}{(x^2+16)^3}$$

x		$-4\sqrt{3}$		-4		0		4		$4\sqrt{3}$	
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$											



$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{x^2+16} = 0$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{x^2+16} = 0$$



(c) $y = f(x) = \dfrac{1}{x^2-16}$

41

Domain $x^2 - 16 \neq 0$

$\iff x \neq \pm 4$

$\iff$

$(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$$f'(x) = \frac{-2x}{(x^2-16)^2}$$

$$f''(x) = \frac{-2(x^2-16)^2 + 2x \cdot 2(x^2-16) \cdot 2x}{(x^2-16)^4}$$

$$= \frac{-2(x^2-16) + 8x^2}{(x^2-16)^3}$$

$$= \frac{6x^2+32}{(x^2-16)^3}$$

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x		-4		0		4	
$f'(x)$	$+$	X	$+$	0	$-$	X	$-$
$f''(x)$	$+$	X	$-$	$-$	$-$	X	$+$
$f(x)$	↗	X	↗		↘	X	↘

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2-16} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2-16} = 0$$

$$\lim_{x\to-4^-} f(x) = \lim_{x\to-4^-} \frac{1}{x^2-16} = +\infty$$

$$\lim_{x\to-4^+} f(x) = \lim_{x\to-4^+} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^-} f(x) = \lim_{x\to4^-} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^+} f(x) = \lim_{x\to4^+} \frac{1}{x^2-16} = +\infty$$

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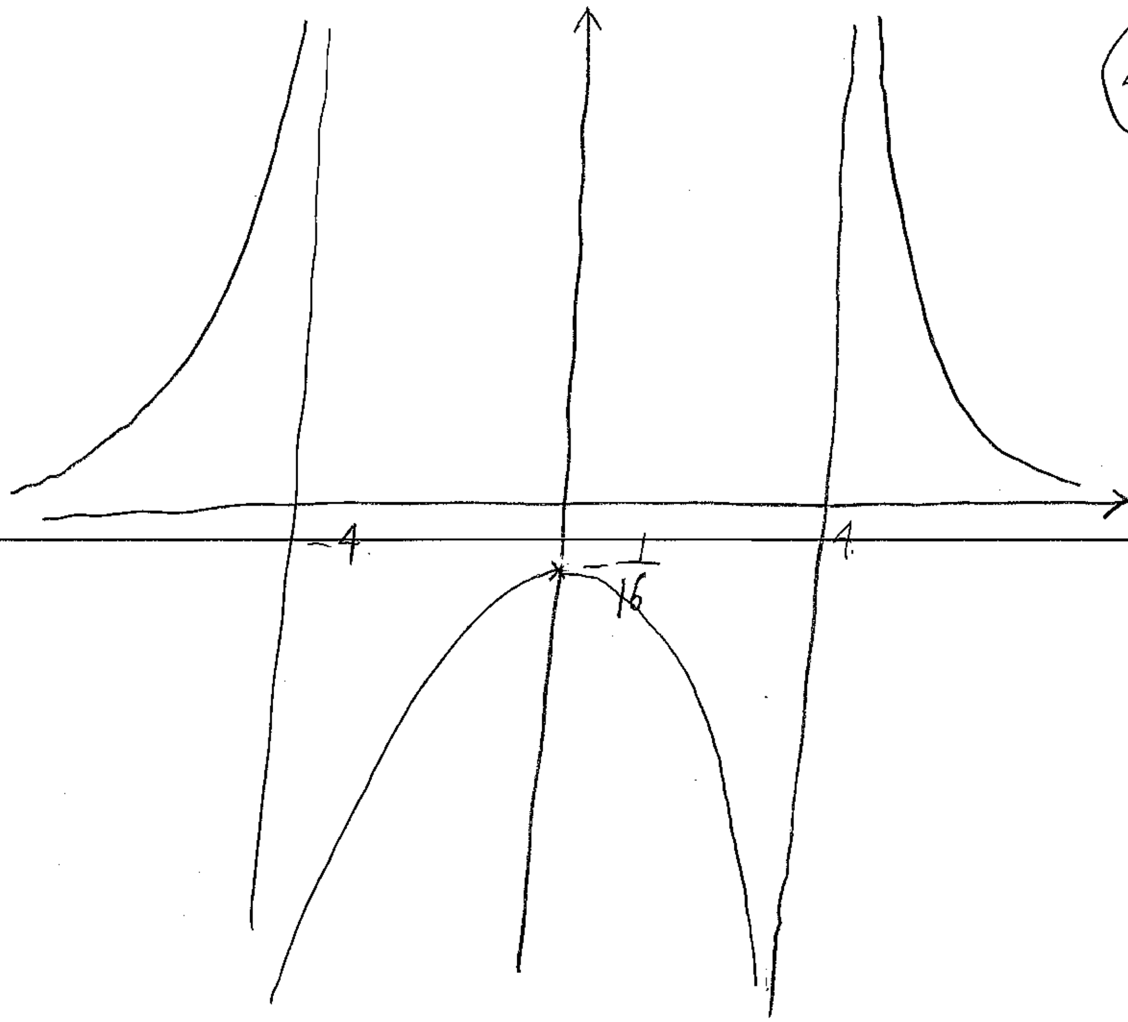


(d) $y = f(x) = \dfrac{x^2}{x^2-16}$

44

Domain $x^2 - 16 \neq 0$

$\Leftrightarrow x \neq \pm 4$

$\Leftrightarrow$

$(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$$f'(x) = \frac{2x(x^2-16) - x^2 \cdot 2x}{(x^2-16)^2}$$

$$= \frac{-32x}{(x^2-16)^2}$$

$$f''(x) = \frac{-32\{1 \cdot (x^2-16)^2 - x \cdot 2(x^2-16) \cdot 2x\}}{(x^2-16)^4}$$

$$= \frac{-32 \cdot \{(x^2-16) - 4x^2\}}{(x^2-16)^3}$$

$$= \frac{32(3x^2+16)}{(x^2-16)^3}$$

x		-4		0		4	
$f'(x)$	$+$	X	$+$	0	$-$	X	$-$
$f''(x)$	$+$	X	$-$	$-$	$-$	X	$+$
$f(x)$	$\nearrow$	X	$\nearrow$	0	$\searrow$	X	$\searrow$

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{x^2}{x^2-16} = +\infty$$

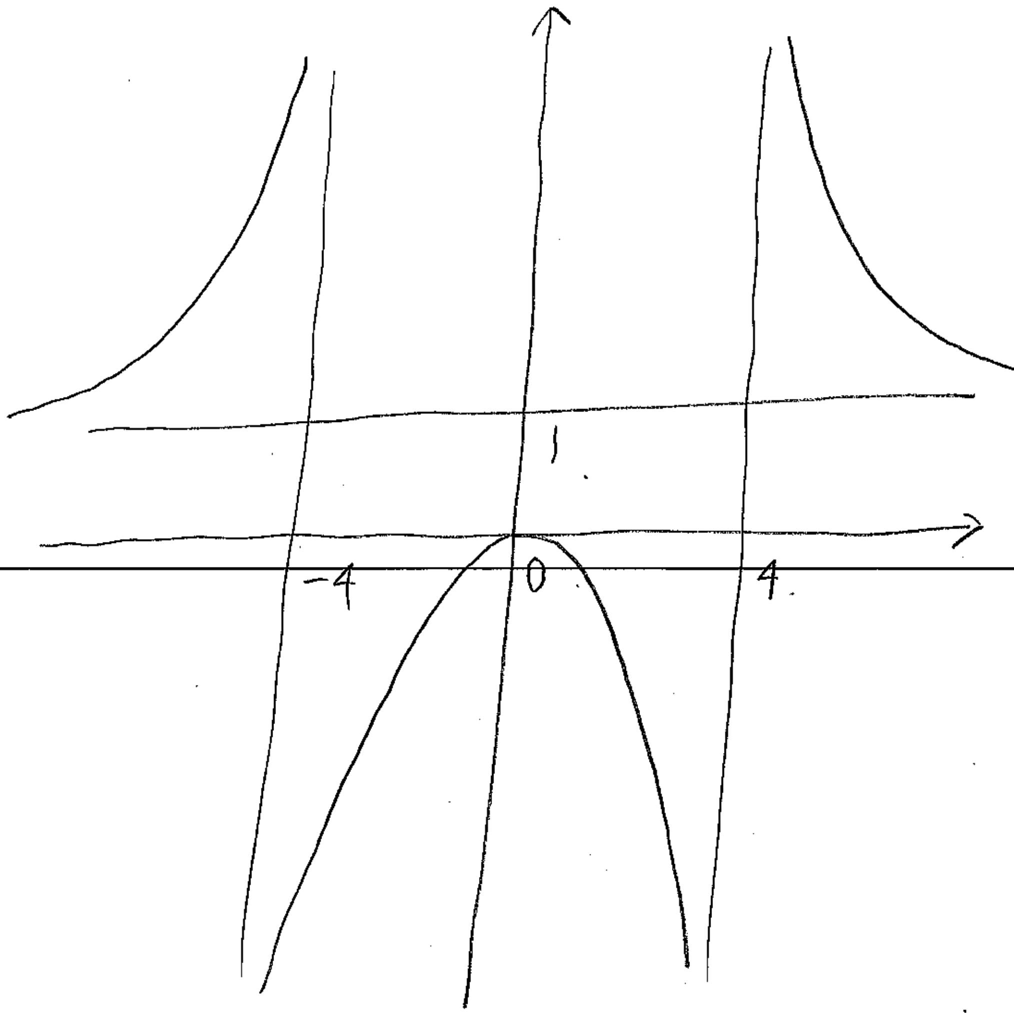
$$\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} \frac{x^2}{x^2-16} = -\infty$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x^2}{x^2-16} = -\infty$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x^2}{x^2-16} = +\infty$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^2}{x^2-16} = 1$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{x^2}{x^2-16} = 1$$



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(e) $y = f(x) = e^{-x} \sin x$ on $[0, 2\pi]$

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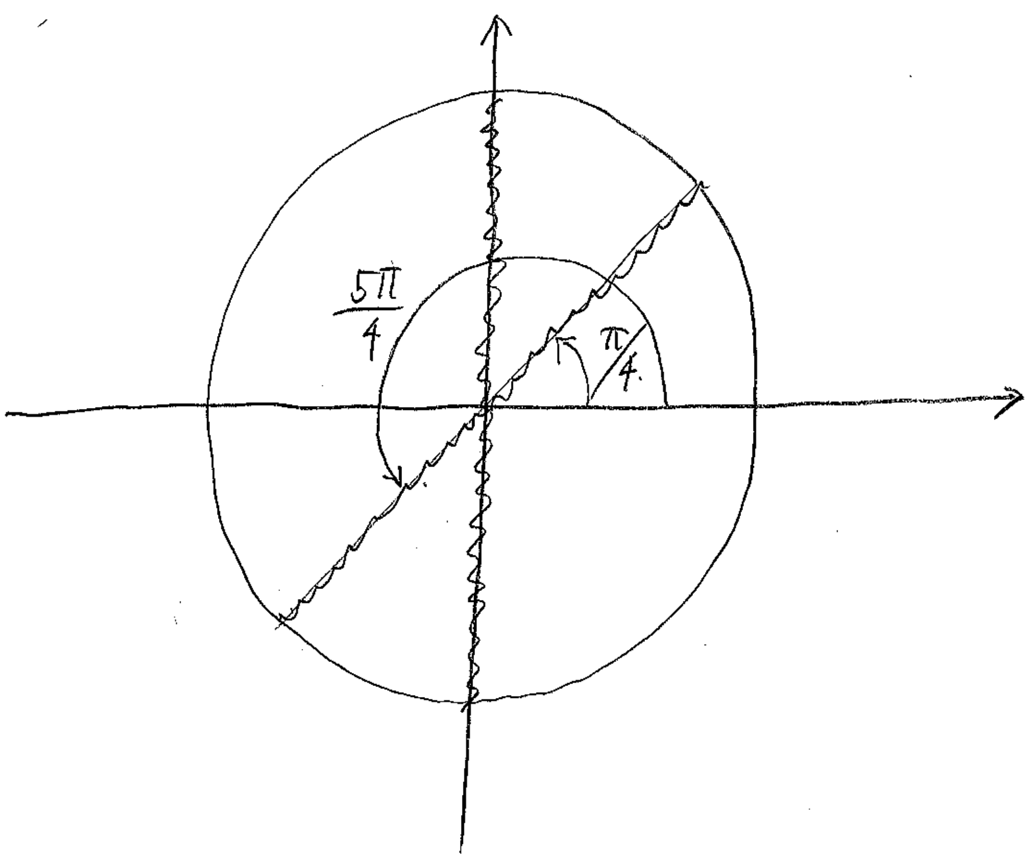
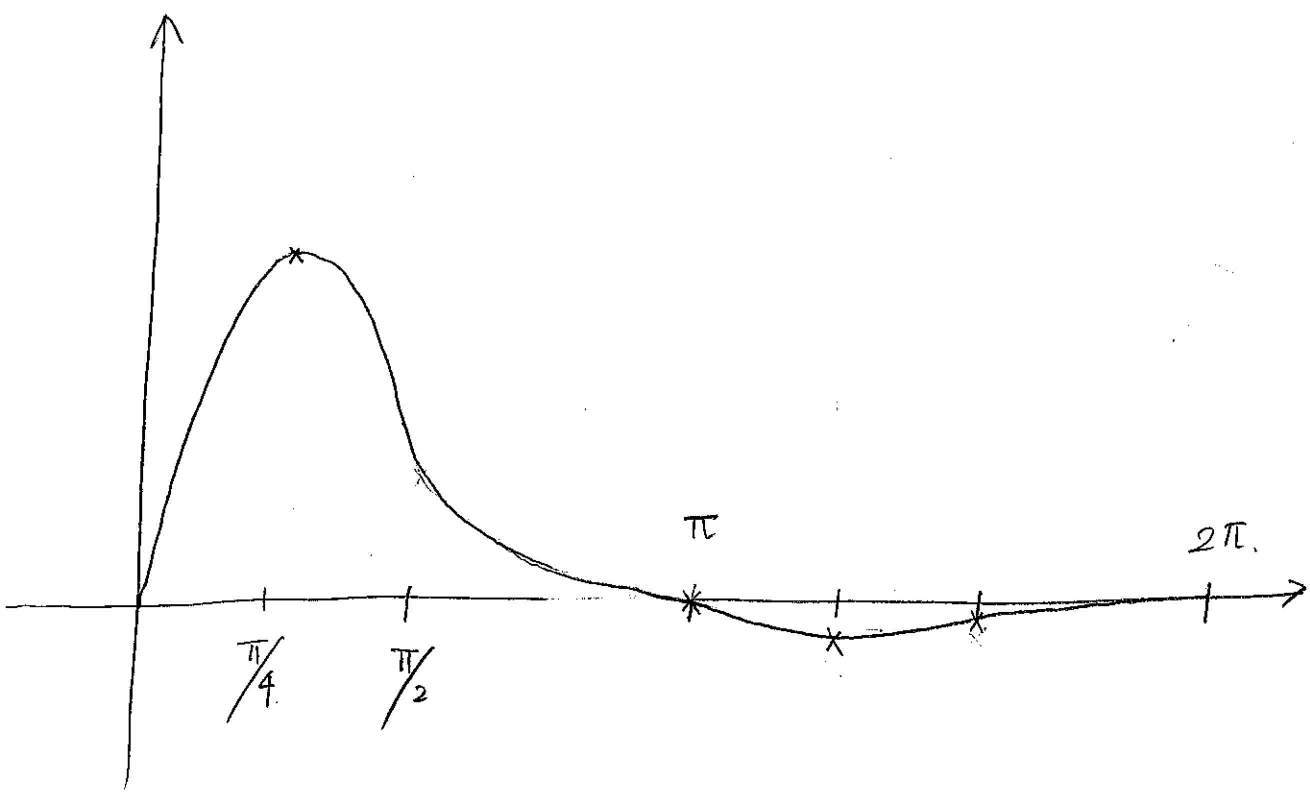
$$f'(x) = -e^{-x} \sin x + e^{-x} \cos x$$
$$= e^{-x} (\cos x - \sin x)$$
$$f''(x) = -e^{-x} (\cos x - \sin x) + e^{-x} (-\sin x - \cos x)$$
$$= -2 e^{-x} \cos x$$



x	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{5\pi}{4}$		$\frac{3\pi}{2}$		2π
$f'(x)$		$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	
$f''(x)$		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}$	$\searrow$	$e^{-\frac{\pi}{2}}$	$\searrow$	$-e^{-\frac{5\pi}{4}}\cdot\frac{\sqrt{2}}{2}$	$\nearrow$	$-e^{-\frac{3\pi}{2}}$	$\nearrow$	0



(f) $y = f(x) = \ln(x^2 - 10x + 24)$

Domain $x^2 - 10x + 24 > 0$ (49)

$$\overset{\shortparallel}{(x-4)(x-6)}$$

$\Leftrightarrow$

$x < 4$ or $6 < x$

$\Leftrightarrow$

$$(-\infty, 4) \cup (6, \infty)$$

$$f'(x) = \frac{2x - 10}{x^2 - 10x + 24}$$

$$= \frac{2(x-5)}{x^2 - 10x + 24}$$

$+24$

$$f''(x) = \frac{2\{1 \cdot (x^2 - 10x + 24) - (x-5)(2x-10)\}}{(x^2 - 10x + 24)^2}$$

$$= \frac{-2(x^2 - 10x + 26)}{(x^2 - 10x + 24)^2}$$

$$= \frac{-2\{(x-5)^2 + 1\}}{(x^2 - 10x + 24)^2}$$

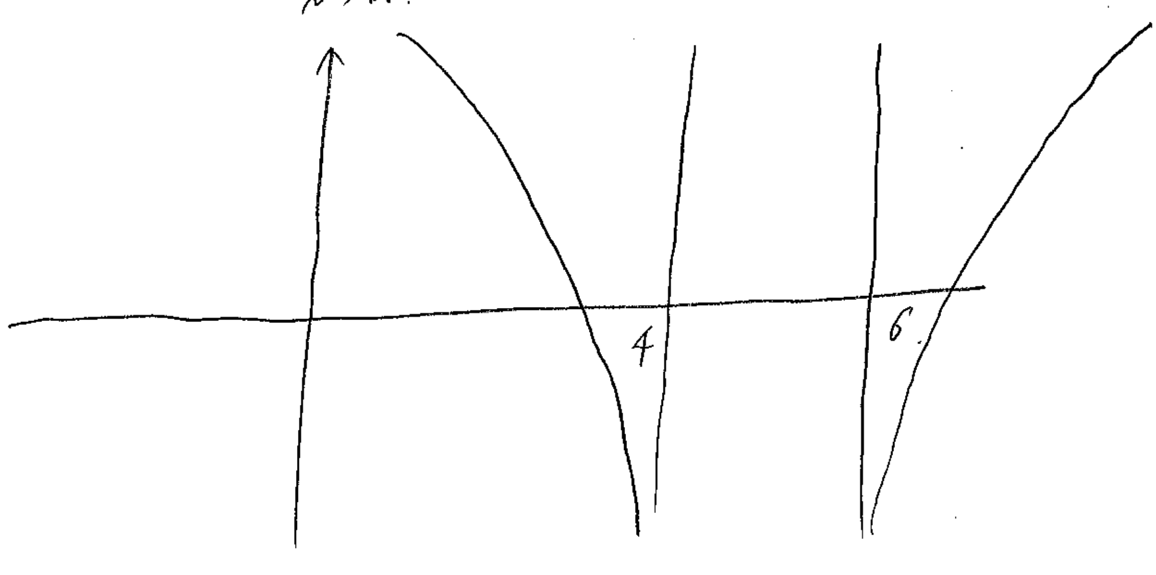
x		4		6	
$f'(x)$	$-$				$+$
$f''(x)$	$-$				$-$
$f(x)$	↘				↗

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \ln(x^2 - 10x + 24) = -\infty$$

$$\lim_{x \to 6^+} f(x) = \lim_{x \to 6^+} \ln(x^2 - 10x + 24) = -\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \ln(x^2 - 10x + 24) = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \ln(x^2 - 10x + 24) = +\infty$$



(8) $y = f(x) = \dfrac{x^3}{x^2+1}$

Domain $(-\infty, \infty)$

$$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2}$$

$$= \frac{x^4 + 3x^2}{(x^2+1)^2}$$

$$= \frac{x^2(x^2+3)}{(x^2+1)^2}$$

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2)2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

$-2x^3+6x$

$4x^3+6x^3+6x-12x^3$

$$= \frac{(4x^3+6x)(x^2+1) - (x^4+3x^2)4x}{(x^2+1)^3}$$

$$= \frac{-2x^3+6x}{(x^2+1)^3}$$

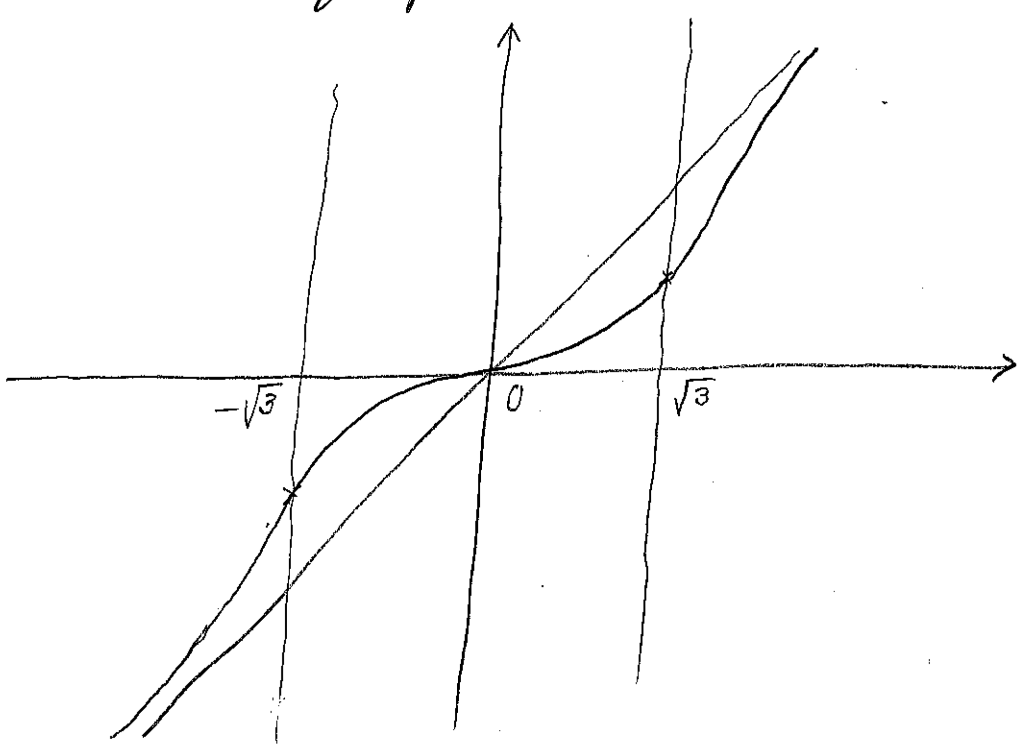
$$= \frac{-2x(x^2-3)}{(x^2+1)^3}$$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$f'(x)$	$+$	$+$	$+$	0	$+$	$+$	$+$
$f''(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$		inf.		inf.		inf.	

$$\frac{x^3}{x^2+1} = x + \frac{-x}{x^2+1}$$

$$\begin{array}{r} x \\ x^2+1 \overline{) \, x^3 } \\ \underline{x^3 \quad +x} \\ -x \end{array}$$

slant asymptote $y = x$



9.1.

$$y = f(x) = \frac{2x^2}{x^2-1} = \frac{2x^2}{(x-1)(x+1)}$$

$$\lim_{x\to -1^-} f(x) = +\infty \qquad \lim_{x\to 1^-} f(x) = -\infty$$

$$\lim_{x\to -1^+} f(x) = -\infty \qquad \lim_{x\to 1^+} f(x) = +\infty$$

vertical asymptotes

$$x = -1, \quad x = 1.$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{2x^2}{x^2-1} = 2.$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{2x^2}{x^2-1} = 2$$

horizontal asymptotes

$$y = 2.$$

9.2

$$y = f(x) = \ln\left(\frac{x^2+2x+3}{x^2-6x+8}\right)$$

$$x^2+2x+3 = (x+1)^2 + 2 > 0$$

$$x^2-6x+8 = (x-2)(x-4)$$

For $f(x)$ to be defined,

$$\frac{x^2+2x+3}{x^2-6x+8} > 0$$

$\Leftrightarrow$

$$x < 2 \ , \quad 4 < x$$

Domain $(-\infty, 2) \cup (4, \infty)$

$$\lim_{x\to 2^-} f(x) = +\infty \qquad \lim_{x\to 4^+} f(x) = +\infty$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \ln\left(\underbrace{\frac{x^2+2x+3}{x^2-6x+8}}_{\to 1}\right) = 0$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \ln\left(\underbrace{\frac{x^2+2x+3}{x^2-6x+8}}_{\to 1}\right) = 0$$

vertical asymptotes $x = 2, \ x = 4$

horizontal asymptotes $y = 0$

10.1

$$f(x) = \frac{-3x^3 + 2x^2 + 7x - 5}{x^2 + x + 1}$$

$$\begin{array}{r|l}
 & -3x + 5 \\
\hline
x^2 + x + 1 & -3x^3 + 2x^2 + 7x - 5 \\
 & -3x^3 - 3x^2 - 3x \\
\hline
 & 5x^2 + 10x - 5 \\
 & 5x^2 + 5x + 5 \\
\hline
 & 5x - 10
\end{array}$$

$$= -3x + 5 + \frac{5x - 10}{x^2 + x + 1}$$

$$\lim_{x \to +\infty} \frac{5x - 10}{x^2 + x + 1} = 0$$

$$\lim_{x \to -\infty} \frac{5x - 10}{x^2 + x + 1} = 0$$

Slant asymptote

$$y = -3x + 5$$

10.2

$$y = f(x) = \sqrt{x^2+2x-3}$$ 56

$$= \sqrt{(x+1)^2-4}$$

$$= \sqrt{(x+1)^2\left\{1-\frac{4}{(x+1)^2}\right\}}$$

$(x >> 0)$

$$= (x+1)\sqrt{1-\frac{4}{(x+1)^2}}$$

$(x << 0)$

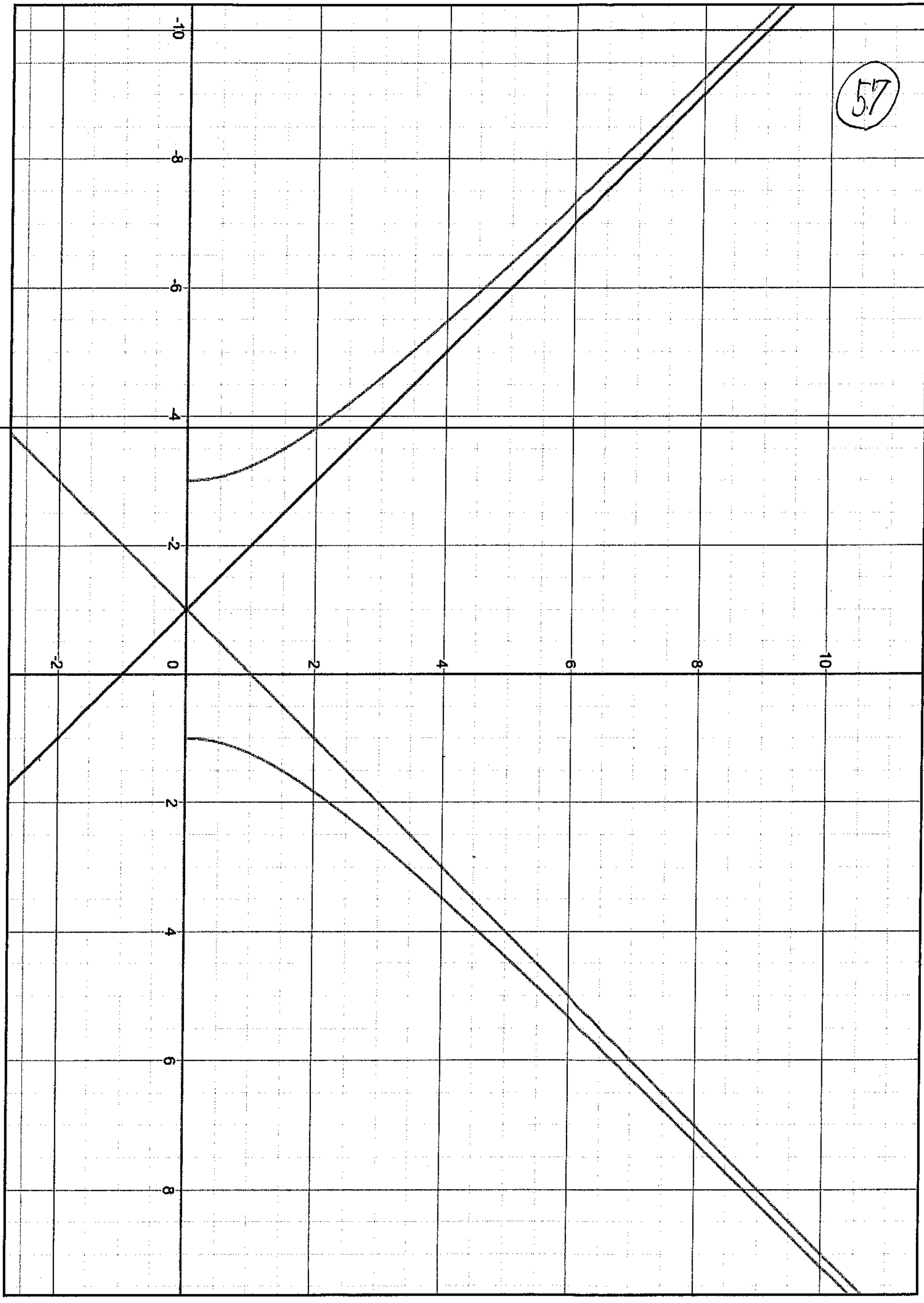
$$= -(x+1)\sqrt{1-\frac{4}{(x+1)^2}}$$

Slant asymptotes

$$y = x+1$$

&

$$y = -(x+1) = -x+1$$


-2
0
2
4
6
8
10
-10
-8
-6
-4
-2
2
4
6
8
57

Check.

$$\lim_{x \to +\infty} \left\{ \sqrt{x^2+2x-3} - (x+1) \right\}$$

$$= \lim_{x \to +\infty} \frac{\left\{ \sqrt{x^2+2x-3} - (x+1) \right\}\left\{ \sqrt{x^2+2x-3} + (x+1) \right\}}{\left\{ \sqrt{x^2+2x-3} + (x+1) \right\}}$$

$$= \lim_{x \to +\infty} \frac{(x^2+2x-3) - (x+1)^2}{\sqrt{x^2+2x-3} + (x+1)}$$

$$= \lim_{x \to +\infty} \frac{-4}{\sqrt{x^2+2x-3} + (x+1)} = 0$$

$$\lim_{x \to -\infty} \left\{ \sqrt{x^2+2x-3} - [-(x+1)] \right\}$$

$$= \lim_{x \to -\infty} \frac{\left\{ \sqrt{x^2+2x-3} - [-(x+1)] \right\}\left\{ \sqrt{x^2+2x-3} + [-(x+1)] \right\}}{\left\{ \sqrt{x^2+2x-3} + [-(x+1)] \right\}}$$

$$= \lim_{x \to -\infty} \frac{(x^2+2x-3) - [-(x+1)]^2}{\left\{ \sqrt{x^2+2x-3} + [-(x+1)] \right\}}$$

$$= \lim_{x \to -\infty} \frac{-4}{\left\{ \sqrt{x^2+2x-3} + [-(x+1)] \right\}} = 0$$

10.3.

$$y = f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}$$

$$= \frac{\cancel{(x-2)}(2x^2+5)}{\cancel{(x-2)}(x+3)}$$

$$\left(\begin{array}{r|l} & \;\;2x - 6 \\ \hline x+3 & 2x^2 \qquad + 5 \\ & 2x^2 + 6x \\ \hline & \quad -6x + 5 \\ & \quad -6x - 18 \\ \hline & \qquad\; 23 \end{array} \right)$$

$$= 2x - 6 + \frac{23}{x+3}$$

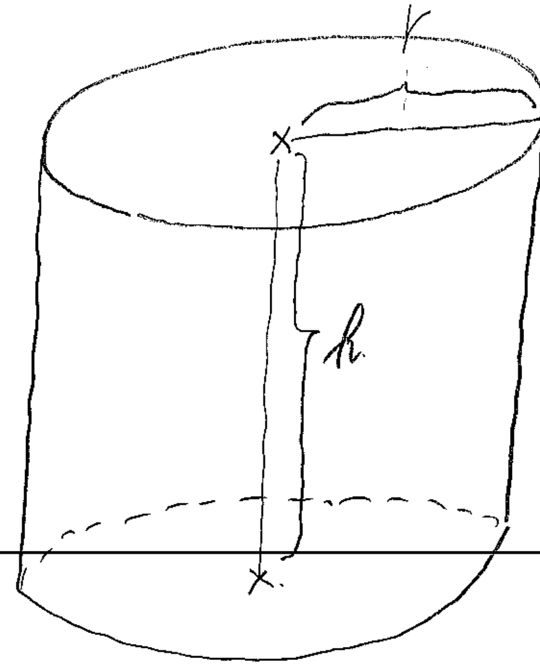
vertical asymptote $x = -3$

horizontal asymptote NONE.

slant asymptote $y = 2x - 6$

11.1.

Picture



Condition

$$\pi r^2 h = 1000$$

Objective

Minimize

$$A = 2 \cdot \pi r^2 + 2\pi r \cdot h$$

$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$(0 < r)$$

Solution

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

$$= \frac{4\pi r^3 - 2000}{r^2}$$

$$= \frac{4\pi \left(r^3 - \frac{500}{\pi} \right)}{r^2}$$

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r	0		$\sqrt[3]{\frac{500}{\pi}}$	
$A'(r)$	X	$-$	0	$+$
$A(r)$	X	$\searrow$	min	$\nearrow$

$\sqrt[3]{\frac{4000}{\pi}}$

22 -

When $r = \sqrt[3]{\frac{500}{\pi}}$

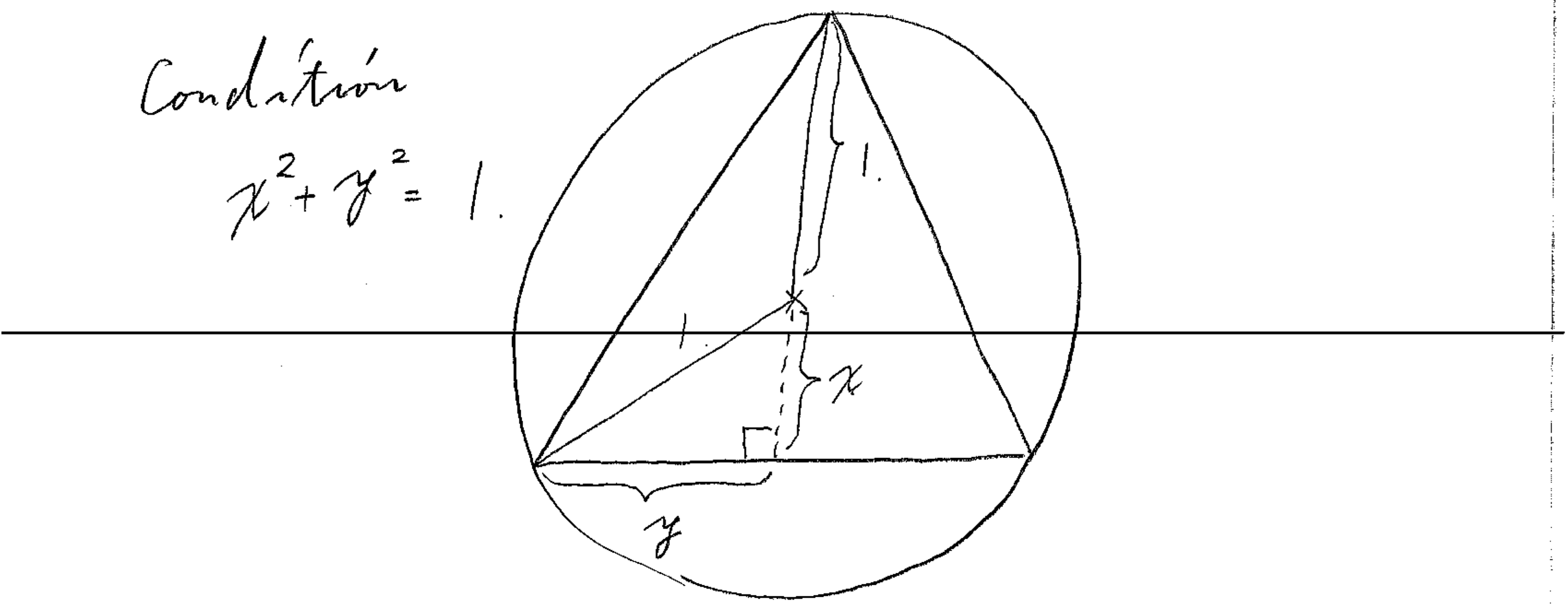
and hence

$$h = \frac{1000}{\pi r^2} = \sqrt[3]{\frac{4000}{\pi}},$$

the surface area A is min.
(and hence the cost)

11.2.

Picture



Condition

$x^2 + y^2 = 1.$

Objective

Maximize

$$A = \frac{1}{2} \cdot 2y(1+x) = y(1+x)$$

$$A(x) = \sqrt{1-x^2}\,(1+x)$$

$$= (1+x)\sqrt{1-x^2}$$

$$0 \leq x \leq 1.$$

$$(\text{or } -1 \leq x \leq 1)$$

Solution

$$A'(x) = 1 \cdot \sqrt{1-x^2} + (1+x)\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$$

$$= \frac{1-x^2-x-x^2}{\sqrt{1-x^2}}$$

$$= \frac{-(2x^2+x-1)}{\sqrt{1-x^2}}$$

$$= \frac{-(2x-1)(x+1)}{\sqrt{1-x^2}}$$

x	0		$1/2$		1
$A'(x)$		$+$	0	$-$	
$A(x)$		$\nearrow$		$\searrow$	

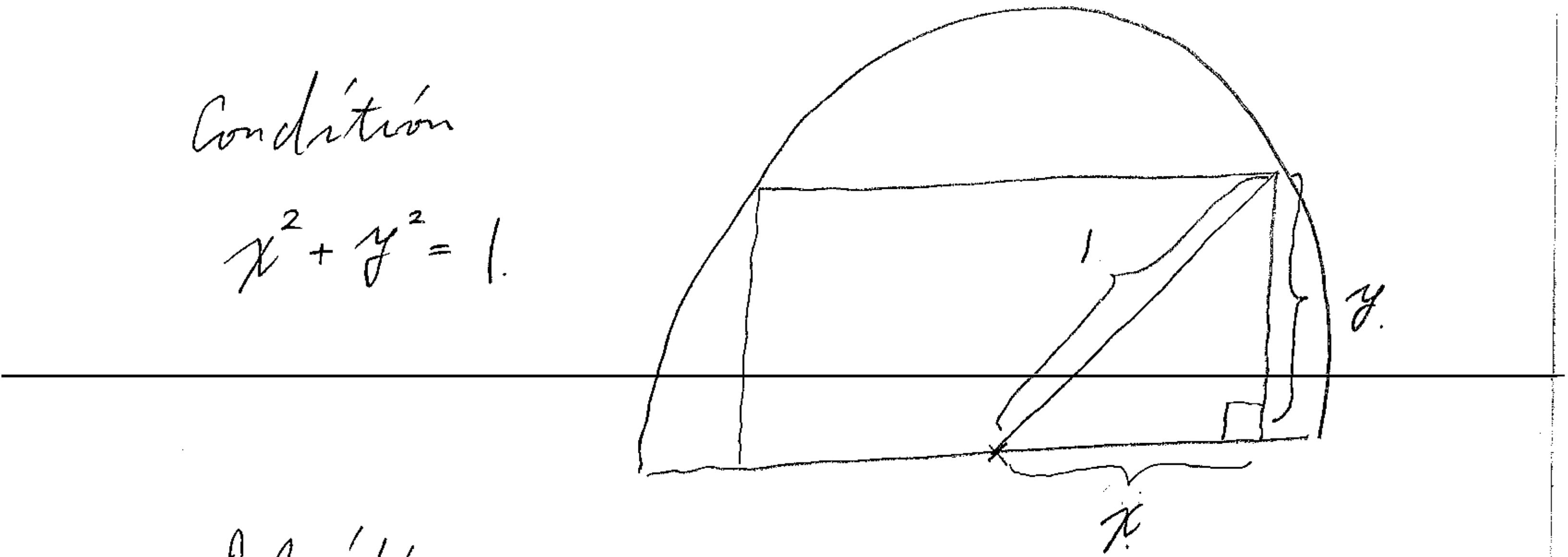
When $x = \frac{1}{2}$,

$$A\left(\frac{1}{2}\right) = \sqrt{1-\left(\frac{1}{2}\right)^2}\left(1+\frac{1}{2}\right)$$

$$= \sqrt{\frac{3}{4}} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{4} \text{ is max.}$$

Note : When $x = \frac{1}{2}$,
the isosceles triangle with the largest area is an equilateral triangle.

Picture



Condition

$$x^2 + y^2 = 1.$$

Objective

Maximize

$$A = 2x \cdot y$$

$$A(x) = 2x\sqrt{1-x^2}$$

$$0 \leq x \leq 1$$

Solution

$$A'(x) = 2\left\{1 \cdot \sqrt{1-x^2} + x\,\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right\}$$

$$= 2 \cdot \frac{1-x^2-x^2}{\sqrt{1-x^2}}$$

$$= 2 \cdot \frac{1 - 2x^2}{1 - x^2}$$

(66)

x	0		$\frac{\sqrt{2}}{2}$		1
$A'(x)$		$+$	0	$-$	
$A(x)$		$\nearrow$		$\searrow$	

When $x = \frac{\sqrt{2}}{2}$,

the rectange has the largest area

$$A\left(\frac{\sqrt{2}}{2}\right) = 2 \cdot \frac{\sqrt{2}}{2} \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2}$$

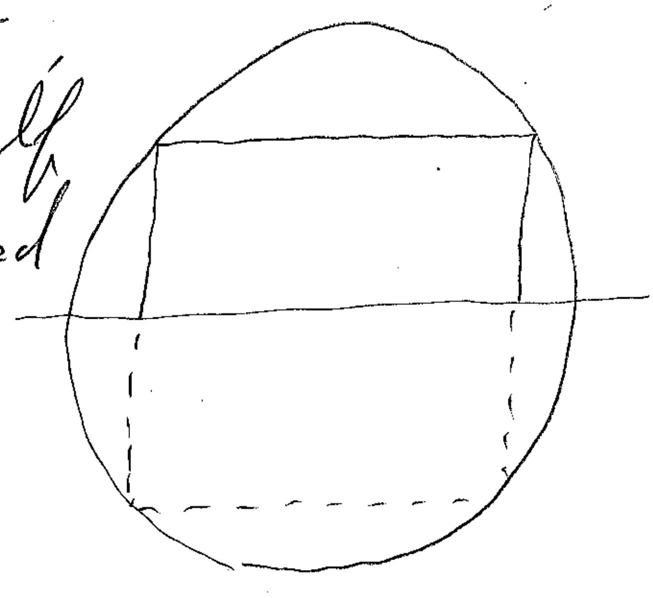
$$= \sqrt{2} \sqrt{\frac{3}{4}} = \frac{\sqrt{6}}{2}$$

Note: When $x = \frac{\sqrt{2}}{2}$

the rectangle is the half

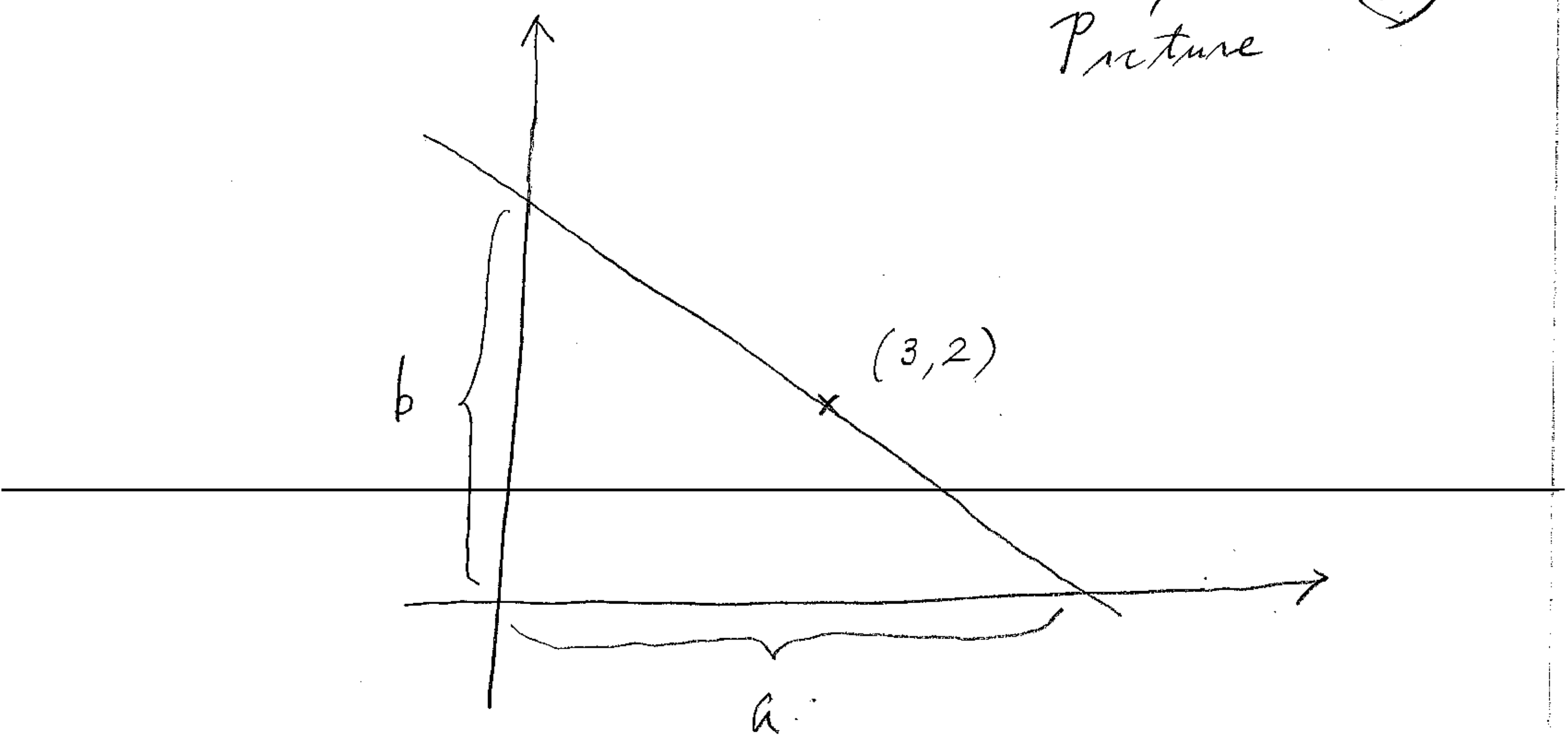
of the square inscribed

in the entire circle



11.4.

Picture



Condition

eq. of line

$$y = -\frac{b}{a}x + b$$

line passes (3, 2)

$$2 = -\frac{b}{a}3 + b$$

$$= b\left(-\frac{3}{a} + 1\right)$$

$$\longrightarrow \quad b = \frac{2}{-\frac{3}{a} + 1}$$

Objective

Minimize

$$A = \frac{1}{2} a b$$

$$A(a) = \frac{1}{\cancel{2}} a \cdot \frac{\cancel{2}}{-\frac{3}{a} + 1}$$

$$= \frac{a^2}{a-3} \qquad (3 < a)$$

Solution

$$A'(a) = \frac{2a(a-3) - a^2 \cdot 1}{(a-3)^2}$$

$$= \frac{a^2 - 6a}{(a-3)^2}$$

$$= \frac{a(a-6)}{(a-3)^2}$$

a	3		6	
$A'(a)$		$-$	0	$+$
$A(a)$		$\searrow$		$\nearrow$

When $a = 6$ $\left(\rightarrow b = \dfrac{2}{-\frac{3}{6}+1} = 4 \right)$

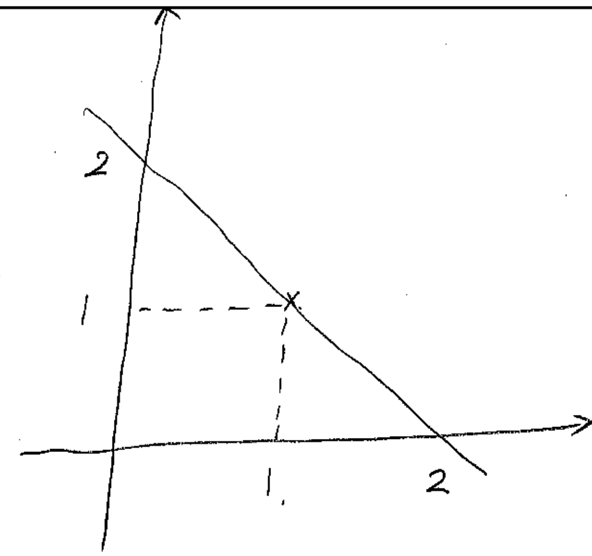
the line cuts off the least area

The equation of the line is

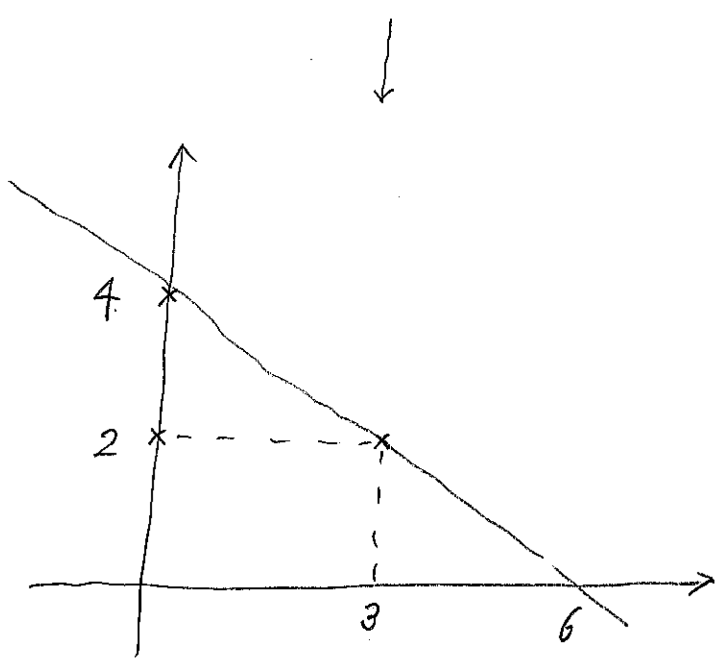
$$y = -\frac{2}{3}x + 4$$

Note :

The line passing $(1,1)$ cuts off the least amount of area, when it passes $(0,2)$ & $(2,0)$.

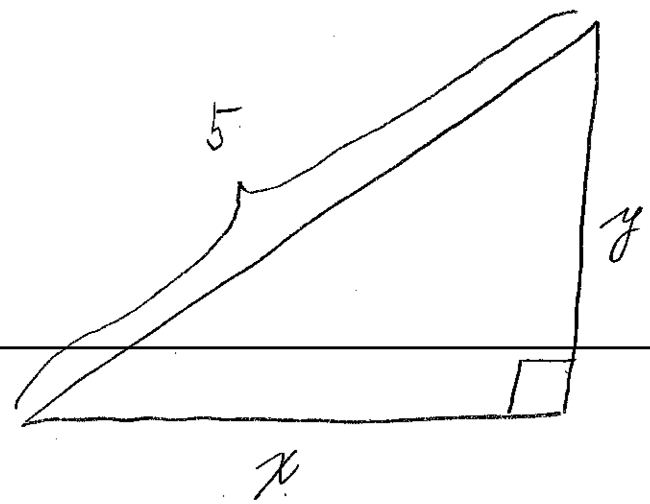


When you stretch this picture vertically $\times 2$ horizontally $\times 3$, you get the picture for the problem.



11.5.

Picture



Condition $\quad x^2 + y^2 = 5^2$

Objective

Maximize

$$A = \frac{1}{2} x y$$

$$A(x) = \frac{1}{2} x \sqrt{5^2 - x^2}$$

$$0 < x < 5$$

Solution

$$A'(x) = \frac{1}{2} \left\{ 1 \cdot \sqrt{5^2 - x^2} + x \frac{-\cancel{2}x}{\cancel{2}\sqrt{5^2 - x^2}} \right\}$$

$$= \frac{1}{2} \cdot \frac{5^2 - x^2 - x^2}{\sqrt{5^2 - x^2}}$$

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$$= \frac{1}{2} \cdot \frac{5^2 - 2x^2}{\sqrt{5^2 - x^2}}$$

x	0		$\frac{5\sqrt{2}}{2}$		5
$A'(x)$		$+$	0	$-$	
$A(x)$		$\nearrow$	max	$\searrow$	

When $x = \frac{5\sqrt{2}}{2}$,

the area $A\left(\frac{5\sqrt{2}}{2}\right)$

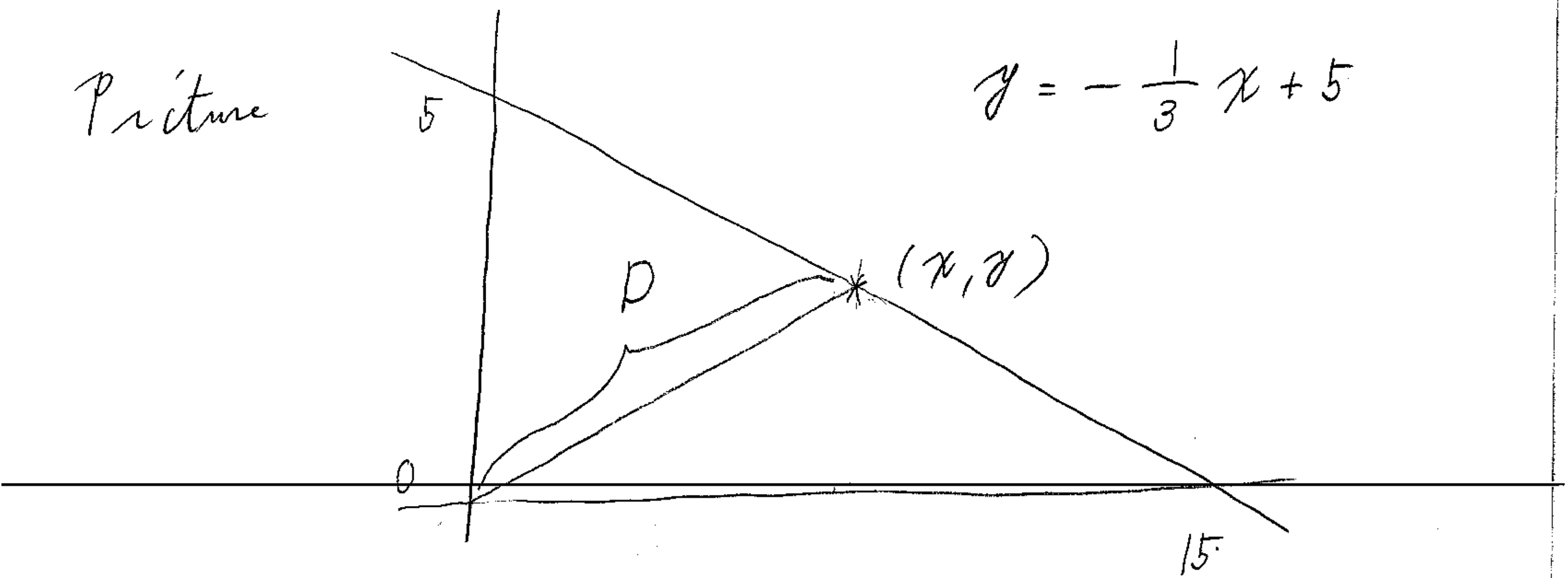
$$= \frac{1}{2} \cdot \frac{5\sqrt{2}}{2} \sqrt{5^2 - \left(\frac{5\sqrt{2}}{2}\right)^2}$$

$$= \frac{1}{2} \frac{5\sqrt{2}}{2} \cdot \frac{5\sqrt{2}}{2} = \frac{25}{4}$$ is max.

Note: The triangle with the largest area is an isosceles right triangle.

11.6

Picture



Condition $y = -\frac{1}{3}x + 5$

Objective

Minimize

$$D = \sqrt{x^2 + y^2}$$

$$D(x) = \sqrt{x^2 + \left(-\frac{1}{3}x + 5\right)^2}$$

$$-\infty < x < \infty$$

Solution

$$D'(x) = \frac{2x + 2\left(-\frac{1}{3}x + 5\right)\left(-\frac{1}{3}\right)}{2\sqrt{x^2 + \left(-\frac{1}{3}x + 5\right)^2}}$$

$$= \frac{\frac{20}{9}x - \frac{10}{3}}{2\sqrt{x^2 + \left(-\frac{1}{3}x + 5\right)^2}}$$

$$= \frac{\frac{20}{9}\left(x - \frac{3}{2}\right)}{2\sqrt{x^2 + \left(-\frac{1}{3}x + 5\right)^2}}$$

---

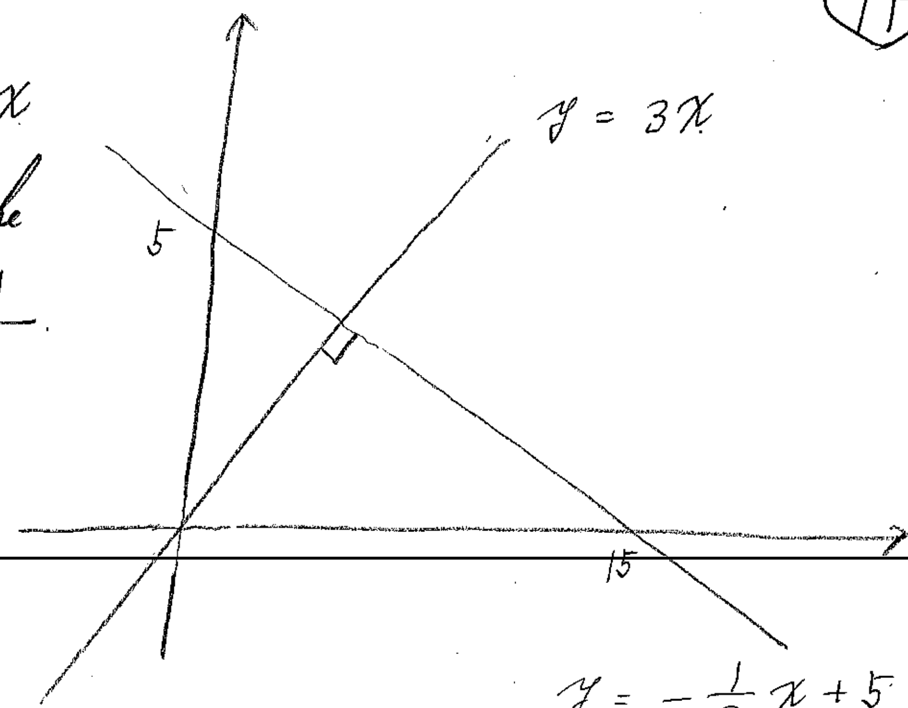
x		$3/2$	
$D'(x)$	$-$	0	$+$
$D(x)$	$\searrow$		$\nearrow$

min.

The point $\left(\frac{3}{2}, \frac{9}{2}\right)$ on the line is the closest to the origin.

Note:

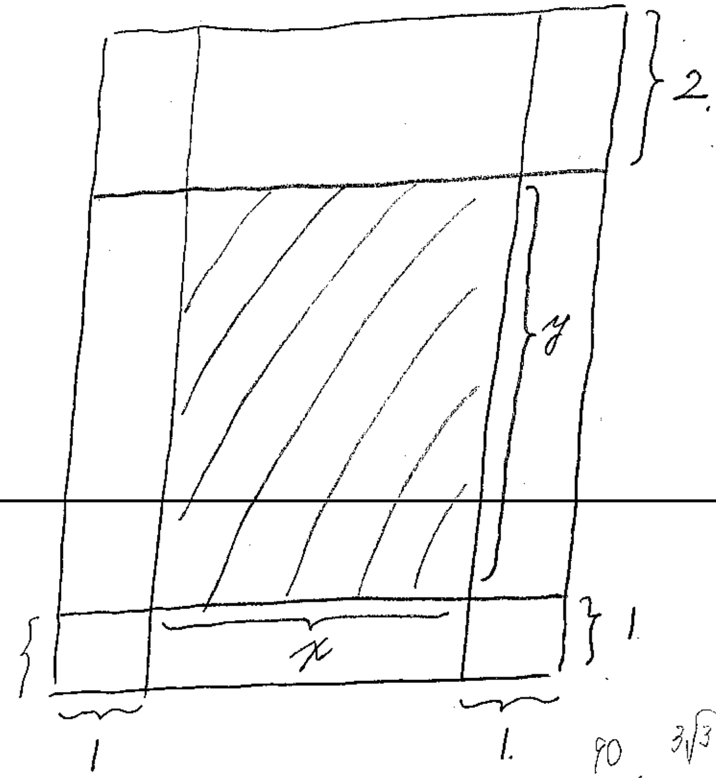
The line $y = 3x$ passing through the origin, and $\perp$ to the line, intersects the line $y = -\frac{1}{3}x + 5$ at $\left(\frac{3}{2}, \frac{9}{2}\right)$



the closest point to the origin.

11.7.

Picture



Condition

$$(x+2)(y+3) = 180$$

Objective

Maximize

$$A = xy$$

$$A(x) = x\left(\frac{180}{x+2} - 3\right)$$

$$0 < x < \frac{180}{3} = 60$$

90 $3\sqrt{3}$ $\cancel{180}$ $\cancel{2}\sqrt{30}$

Solution

$$A'(x) = 1 \cdot \left(\frac{180}{x+2} - 3\right) + x \cdot \left\{-\frac{180}{(x+2)^2}\right\}$$

$$= \frac{180(x+2) - 3(x+2)^2 - 180x}{(x+2)^2}$$

$$= \frac{-3x^2 - 12x + 348}{(x+2)^2}$$

$$= \frac{-3(x^2 + 4x - 116)}{(x+2)^2}$$

$$= \frac{-3\{x - (-2 + 2\sqrt{30})\}\{x - (-2 - 2\sqrt{30})\}}{(x+2)^2}$$

x	0		$-2+2\sqrt{30}$	
$A'(x)$		$+$	0	$-$
$A(x)$		$\nearrow$	max.	$\searrow$

When $x = -2 + 2\sqrt{30}$.

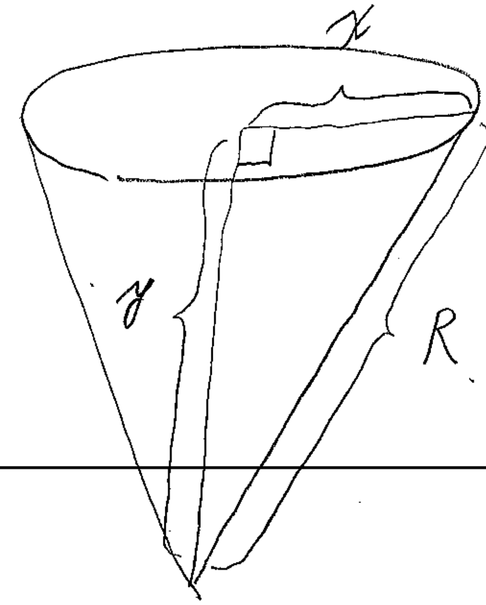
$\rightarrow y = 3\sqrt{30} - 3$,

the printing area

is the largest.

11.8

Picture



Condition

$$x^2 + y^2 = R^2$$

Objective

Maximize

$$V = \frac{1}{3}\pi x^2 \cdot y$$

$$V(x) = \frac{1}{3}\pi x^2 \sqrt{R^2 - x^2}$$

$$0 \leqq x \leqq R$$

Solution

$$V'(x) = \frac{1}{3}\pi \times$$

$$\left\{ 2x\sqrt{R^2 - x^2} + x^2 \frac{-\cancel{2}x}{\cancel{2}\sqrt{R^2 - x^2}} \right\}$$

$$= \frac{1}{3}\pi \cdot \frac{x\{2(R^2-x^2)-x^2\}}{\sqrt{R^2-x^2}}$$

$$= \frac{1}{3}\pi \frac{x\{2R^2-3x^2\}}{\sqrt{R^2-x^2}}$$

x	0		$\sqrt{\frac{2}{3}}R$		R
$V'(x)$		$+$	0	$-$	
$V(x)$		$\nearrow$		$\searrow$	

When $x = \sqrt{\frac{2}{3}}R = \frac{\sqrt{6}}{3}R$,

$$V\left(\sqrt{\frac{2}{3}}R\right) = \frac{1}{3}\pi\left(\sqrt{\frac{2}{3}}R\right)^2\sqrt{R^2-\left(\sqrt{\frac{2}{3}}R\right)^2}$$

$$= \frac{1}{3}\pi \cdot \frac{2}{3}R^2 \cdot \sqrt{\frac{1}{3}}R$$

$$= \frac{2\sqrt{3}}{27}\pi R^3$$

is the max.