Study Guide for Exam 3

1. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$ and the values on the critical value(s) $f(c)('s)$. You should know what the definition of a critical value is.

Example Problems:

1.1. Find the absolute maximum/minimum and local maximum/minimum of the function defined by

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on the closed interval $[-1, 4]$.

1.2. Find the absolute maximum and absolute minimum values of the function f on the given interval.

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$
(b) $f(x) == xe^{x/2}$ on $[-3, 1]$
(c) $f(x) = (x^2 - 1)^3$ on $[-1, 3]$
(d) $f(t) = 2\cos t + \sin 2t$ on $[0, 2\pi]$
(e) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$
(f) $f(x) = x - \sqrt{x}$ on $[0, 9]$

2. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

Example Problems:

2.1. The first derivative of a function f is given by

$$f'(x) = (x + 2)^2(x + 1)(x - 1)^3(x - 3)^2(x - 5).$$

Find the values of x for which the function f takes
(a) local maximum, and
(b) local minimum.

2.2.
(a) Find the critical numbers of the function $f(x) = x^8(x - 4)^7$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you that the Second Derivative test does not?

3. You are supposed to tell whether the graph of a function is concave up/down, and find its inflection points, by looking at the second derivative of the function.

Example Problems:

3.1. Determine how the concavity changes for the function

$$f(x) = \frac{1}{2}x - \sin(x)$$

on the interval $(0, 3\pi)$.

3.2. The second derivative of a function f is given by

$$f''(x) = (x+5)^3(x+2)^2(x-2)^5(x-3)^3(x-6)^2.$$

find the x-coordinates of all the inflection points.

3.3. How many inflection points does the graph of the function $y = f(x) = x^5 - 5x^4 + 25x$ have ?

3.4. We have a function whose first derivative is given by the formula $f'(x) = (x-1)^2(x+3)^3$. Find the local extrema and the inflection points of the function.

4. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.

Example Problems:

4.1.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$

(b) $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$

(c) $\lim_{x\to 0} \frac{e^{7x} - \cos 2x}{\tan(3x)}$

(d) $\lim_{x\to 0} \frac{\sin x}{1-x^2}$

(e) $\lim_{x\to 0} \frac{3x - \sin(3x)}{3x - \tan(3x)}$

(f) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

5. You are suppose to know how to compute the limits of the form $\pm\infty \times 0, \infty - \infty$.

Example Problems:

5.1. Compute the following limits:

(a) $\lim_{x\to 0^+} \sin(x)\ln(2x)$

(b) $\lim_{x\to\infty} 2x\tan\left(\dfrac{1}{3x}\right)$

(c) $\lim_{x\to\left(\frac{\pi}{2}\right)^-} (2x-\pi)\cdot\tan(x)$

(d) $\lim_{x\to 1}\left(\dfrac{x}{x-1} - \dfrac{1}{\ln(x)}\right)$

(e) $\lim_{x\to 4}\left(\dfrac{1}{\sqrt{x}-2} - \dfrac{4}{x-4}\right)$

(f) $\lim_{x\to 0} x^2\tan\left(\dfrac{1}{5x^2+2}\right)$ $\quad$ (f^*) $\lim_{x\to\infty} x^2\tan\left(\dfrac{1}{5x^2+2}\right)$

6. You are supposed to be able to compute the limits $\lim\limits_{x\to a}[f(x)]^{g(x)}$ of the form $0^0, \infty^0, 1^\infty$.

Example Problems:

6.1. Compute the following limits:

(a) $\lim_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{7x}$

(b) $\lim_{x\to\infty}\left(\dfrac{x+3}{x-2}\right)^{4x+1}$

(c) $\lim_{x\to\infty}(2x+e^{5x})^{1/x}$

(d) $\lim_{x\to 0^+}\tan(5x)^{\sin x}$

7. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply this corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise: If $f'(x) = 0$ for all values of $x \in (a, b)$, then a continuous function f on the closed interval $[a, b]$ is actually a constant.

Example Problems:

7.1. Consider the function $f(x) = x^3 - x$ over the interval $[0, 2]$. Does it satisfy the conditions for the Mean Value Theorem to hold ? If it does, find the value(s) $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}.$$

7.2. Consider the function $y = f(x) = x^{2/3}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem to hold ? Do we have any value $c \in (-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} \ ?$$

7.3. Consider the function $y = f(x) = x^{1/5}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem to hold ? Do we have any value $c \in (-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}?$$

What can you say about the Mean Value Theorem in this situation ?

7.4. Determine the exact value of

$$\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right).$$

7.5. Determine the exact value of

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(7).$$

7.6. Consider the equation $f(x) = x^3 + x - 1 = 0$.

Determine how many solutions are there for the above equation on the interval $[0, 1]$, using the Intermediate Value Theorem and the Mean Value Theorem.

8. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down), and also by determining the horizontal/vertical asymptotes and x-intercept (y-intercept).

Example Problems:

8.1. Draw the graph of the following function:

(a) $y = f(x) = \dfrac{x}{x^2 - 16}$

(b) $y = f(x) = \dfrac{x}{x^2 + 16}$

(c) $y = f(x) = \dfrac{1}{x^2 - 16}$

(d) $y = f(x) = \dfrac{x^2}{x^2 - 16}$

(e) $y = f(x) = e^{-x} \sin x$ on $[0, 2\pi]$

(f) $y = f(x) = \ln(x^2 - 10x + 24)$

(g) $y = f(x) = \dfrac{x^3}{x^2 + 1}$

9. You are supposed to be able to determine the horizontal/vertical asymptote(s) of the graph of a function.

Example Problems:

9.1. Find the horizontal/vertical asymptote(s) of the graph of the function

$$y = f(x) = \frac{2x^2}{x^2 - 1}.$$

9.2. Find the horizontal/vertical asymptote(s) of the graph of the function

$$y = f(x) = \ln\left(\frac{x^2 + 2x + 3}{x^2 - 6x + 8}\right).$$

10. You are supposed to be able to determine the equation of the slant asymptote of a function.

Example Problems:

10.1. Find the equation of the slant asymptote of the function

$$f(x) = \frac{-3x^3 + 2x^2 + 7x - 5}{x^2 + x + 1}.$$

10.2. Find the equation of the slant asymptote(s) of the function

$$f(x) = \sqrt{x^2 + 2x - 3}.$$

10.3. Find the equations of the horizontal, vertical, and slant asymptotes of the function

$$f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}.$$

11. Total of 3 Optimization Problems will be given in Exam 3.

Of particular importance are:

- Cylinder Problem
- Maximizing or minimizing the area of a figure (rectangle, isosceles, right triangle etc.) under the given restrictions
- Given a line, find the closest point on the line to a fixed point.
- Poster Problem

- Coffee cup problem

Example Problems:

11. 1. A cylindrical can (with a top and bottom) is to be made to hold 1 L $= 1,000$ cm^3 of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

11.2. What is the largest area of an isosceles triangle inscribed in a circle of radius 1 ?

11.3. Find the area of the largest rectangle that can be inscribed in a semi-circle of radius 3.

11.4. Find the equation of a line passing through the point $(3, 2)$, which cuts off the least amount of area from the first quadrant.

11.5. Among all the right triangles whose hypotenuse has length 5, find the area of the one whose area is maximum.

11.6. On the line $y = -\frac{1}{3}x + 5$, find the point that is closest to the origin.

11.7. A poster is to have an area of 180 in^2 with 1-in margins at the bottom and sides and 2-in margin at the top. What dimensions will give the largest printing area ?

11.8. A cone-shaped drinking cup is made from a circular piece of paper of fixed radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.