Answer Keys for Study Guide for Exam 2.

I. I.

(i) $y = \sin(\sin(\sin x))$

$$x \longmapsto \sin x$$
$$\sin x = u \longmapsto \sin u$$
$$\sin u = t \longmapsto \sin t$$
$$\sin t = y$$

$$x \longrightarrow y$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

$$= \cos t \cdot \cos u \cdot \cos x$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$$

(ii) $y = \cos(2\pi \cdot 3^x)$ 2

$$\begin{array}{ccccc} x & \longmapsto & 2\pi \cdot 3^x & & \\ & & \parallel & & \\ & & u & \longmapsto & \cos u \\ & & & & \parallel \\ & & & & y \end{array}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -\sin u \cdot 2\pi \cdot 3^x \cdot \ln 3$$

$$= -\sin(2\pi \cdot 3^x) \cdot 2\pi \cdot 3^x \cdot \ln 3$$

(iii) $y = \left(\frac{t-2}{2t+1}\right)^9$

$$\begin{array}{ccccc} t & \longmapsto & \frac{t-2}{2t+1} & & \\ & & \parallel & & \\ & & u & \longmapsto & u^9 \\ & & & & \parallel \\ & & & & y \end{array}$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

$$= 9u^8 \cdot \frac{1 \cdot (2t+1) - (t-2) \cdot 2}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$$

(iv) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

Observe

$$\left(\sqrt{x + \bigcirc}\right)' = \frac{1 + \bigcirc'}{2\sqrt{x + \bigcirc}}$$

$$\frac{dy}{dx} = \left(\sqrt{x + \sqrt{x + \sqrt{x}}}\right)'$$

$$= \frac{1 + \left(\sqrt{x+\sqrt{x}}\right)'}{2\sqrt{x + \sqrt{x+\sqrt{x}}}} = \frac{1 + \frac{1 + \left(\sqrt{x}\right)'}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{2^2 \cdot \sqrt{x+\sqrt{x}} \cdot \sqrt{x} + 2\sqrt{x} + 1}{2^3 \sqrt{x + \sqrt{x+\sqrt{x}}} \cdot \sqrt{x + \sqrt{x}} \cdot \sqrt{x}}$$

1.1 (iv) revisited

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

Observe

$$\left(\sqrt{\bigcirc}\right)' = \frac{\bigcirc'}{2\sqrt{\bigcirc}}$$

by the chain rule

$$\frac{dy}{dx} = \frac{\left(x + \sqrt{x + \sqrt{x}}\right)'}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{1 + \left(\sqrt{x + \sqrt{x}}\right)'}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{1 + \dfrac{\left(x + \sqrt{x}\right)'}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{1 + \dfrac{1 + \dfrac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

$$= \frac{1 + \dfrac{2\sqrt{x}+1}{2\sqrt{x+\sqrt{x}}\cdot 2\sqrt{x}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

$$= \frac{2^2\cdot\sqrt{x+\sqrt{x}}\cdot\sqrt{x} + 2\sqrt{x}+1}{2^3\sqrt{x+\sqrt{x+\sqrt{x}}}\cdot\sqrt{x+\sqrt{x}}\cdot\sqrt{x}}$$

(iv*)

$$y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[rrrddd, bend right] & \sqrt{x} \arrow[d, equal] & & \\
& u \arrow[r, mapsto] & \sqrt{1+u} \arrow[d, equal] & \\
& & t \arrow[r, mapsto] & \sqrt{1+t} \arrow[d, equal] \\
& & & y.
\end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{1+t}} \cdot \frac{1}{2\sqrt{1+u}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{8\sqrt{1+\sqrt{1+\sqrt{x}}} \cdot \sqrt{1+\sqrt{x}} \cdot \sqrt{x}}$$

(v) $y = e^{\sec 3\theta}$

$$\theta \longmapsto 3\theta = u \longmapsto \sec u = t \longmapsto e^{t} = y$$

$$\frac{dy}{d\theta} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{d\theta}$$

$$= e^{t} \cdot \sec u \tan u \cdot 3$$

$$= e^{\sec 3\theta} \cdot \sec 3\theta \cdot \tan 3\theta \cdot 3$$

(vi) $y = e^{2^{x^3}}$

$$x \to x^{3} = u \longmapsto 2^{u} = t \longmapsto e^{t} = y$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

$$= e^{t} \cdot 2^{u} \cdot \ln 2 \cdot 3x^{2}$$

$$= e^{2^{x^3}} \cdot 2^{x^3} \cdot \ln 2 \cdot 3x^{2}.$$

(vii) $\quad y = \sin^{-1}\left(\frac{1}{x}\right)$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[ddrr, bend right] & \frac{1}{x} \arrow[d, equal] & \\ & u \arrow[r, mapsto] & \sin^{-1}(u) \arrow[d, equal] \\ & & y \end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \left(-\frac{1}{x^2}\right) = -\frac{1}{\sqrt{x^4-x^2}}$$

(viii) $y = \tan^{-1}(\sqrt{x})$ 1.1 (viii)

$$
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[ddrr, bend right] & \sqrt{x} \arrow[d, phantom, "\shortparallel"] & \\
& u \arrow[r, mapsto] & \tan^{-1} u. \arrow[d, phantom, "\shortparallel"] \\
& & y
\end{tikzcd}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{1+u^2} \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{1}{1+x^{\cancel{2}}} \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}\,(1+x^{\cancel{2}})}
\end{aligned}
$$

(ix) $y = \ln|\sec(3\theta) + \tan(3\theta)|$.

$$
\begin{tikzcd}
\theta \arrow[r] \arrow[dddrrr, bend right] & 3\theta \arrow[d, phantom, "\shortparallel"] & & \\
& u \arrow[r, mapsto] & \sec u + \tan u \arrow[d, phantom, "\shortparallel"] & \\
& & t \arrow[r, mapsto] & \ln|t| \arrow[d, phantom, "\shortparallel"] \\
& & & y.
\end{tikzcd}
$$

$$\frac{dy}{d\theta} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{d\theta}$$

$$= \frac{1}{t} \cdot (\sec u \tan u + \sec^2 u) \cdot 3$$

$$= \frac{1}{\cancel{\sec u + \tan u}} \cdot \cancel{(\tan u + \sec u)} \sec u \cdot 3$$

$$= \sec u \cdot 3$$

$$= 3 \sec(3\theta)$$

(x) $$y = \ln\left(e^{\sin x} + e^{-\sin x}\right)$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[rrr, bend right] & \sin x = u \arrow[r, mapsto] & e^u + e^{-u} = t \arrow[r, mapsto] & \ln t = y
\end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{t} \cdot (e^u - e^{-u}) \cdot \cos x$$

$$= \frac{e^{\sin x} - e^{-\sin x}}{e^{\sin x} + e^{-\sin x}} \cdot \cos x$$

(xi) $$y = \frac{e^x}{\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{(e^x)'\sqrt{x^2+1} - e^x \cdot \left(\sqrt{x^2+1}\right)'}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{e^x \cdot \sqrt{x^2+1} - e^x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{e^x\,(x^2+1-x)}{(x^2+1)\sqrt{x^2+1}}$$

(xii) $$y = \ln(\sin(x^2))$$

$$\begin{array}{l} x \longmapsto x^2 \\ \qquad\quad \overset{\shortparallel}{u} \longmapsto \sin u \\ \qquad\qquad\qquad\quad \overset{\shortparallel}{t} \longmapsto \ln t \\ \qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{y} \end{array}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{t} \cdot \cos u \cdot 2x$$

$$= \frac{1}{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$$

1, 2.

$$F(x) = f^{-1}(\{g(x)\}^2)$$ 12

$$F'(x) = (f^{-1})'(\{g(x)\}^2) \cdot 2\,g(x) \cdot g'(x)$$

$\rightarrow$

$$F'(1) = (f^{-1})'(\{g(1)\}^2) \cdot 2\,g(1) \cdot g'(1)$$

$$= (f^{-1})'(\underset{\overset{\shortparallel}{9}}{3^2}) \cdot 2 \cdot 3 \cdot 2$$

(Observe.

$$(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{1}{3}$$

as $f(2) = 9$ & $f^{-1}(9) = 2$.)

$$= \frac{1}{3} \cdot 2 \cdot 3 \cdot 2 = 4.$$

1.3.

13

$$F(x) = [g(f^{-1}(x))]^2$$

$$F'(x) = 2 \cdot g(f^{-1}(x)) \cdot g'(f^{-1}(x)) \cdot (f^{-1})'(x)$$

$$F'(9) = 2 \cdot g(f^{-1}(9)) \cdot g'(f^{-1}(9)) \cdot (f^{-1})'(9)$$

$$\left(\begin{array}{l} f(2) = 9 \rightarrow (f^{-1})(9) = 2 \\ (f^{-1})'(9) = \dfrac{1}{f'(2)} = \dfrac{1}{5} \end{array} \right)$$

$$= 2 \cdot g(2) \cdot g'(2) \cdot \frac{1}{5}$$

$$= 2 \cdot 3 \cdot (-2) \cdot \frac{1}{5}$$

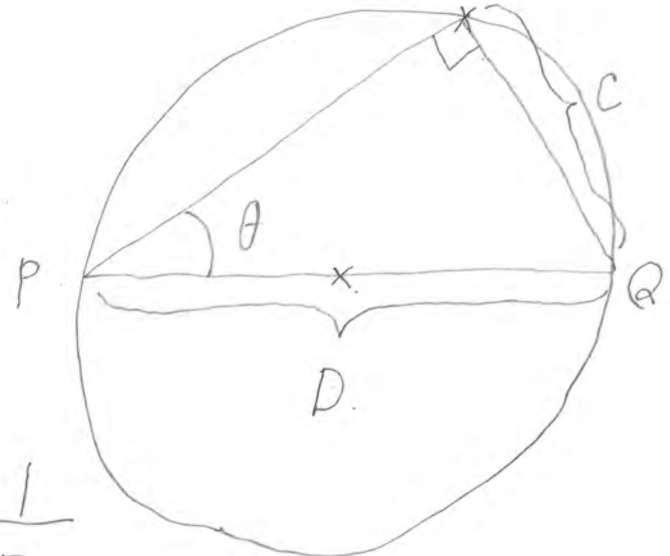
$$= -\frac{12}{5}$$

1.4.

14

$$\sin\theta = \frac{c}{D}$$

$$\rightarrow \quad \theta = \sin^{-1}\left(\frac{c}{D}\right)$$



$$\rightarrow$$

$$\frac{d\theta}{dc} = \frac{1}{\sqrt{1-\left(\frac{c}{D}\right)^2}} \cdot \frac{1}{D}$$

$$= \frac{1}{\sqrt{D^2 - c^2}}$$

1. 5

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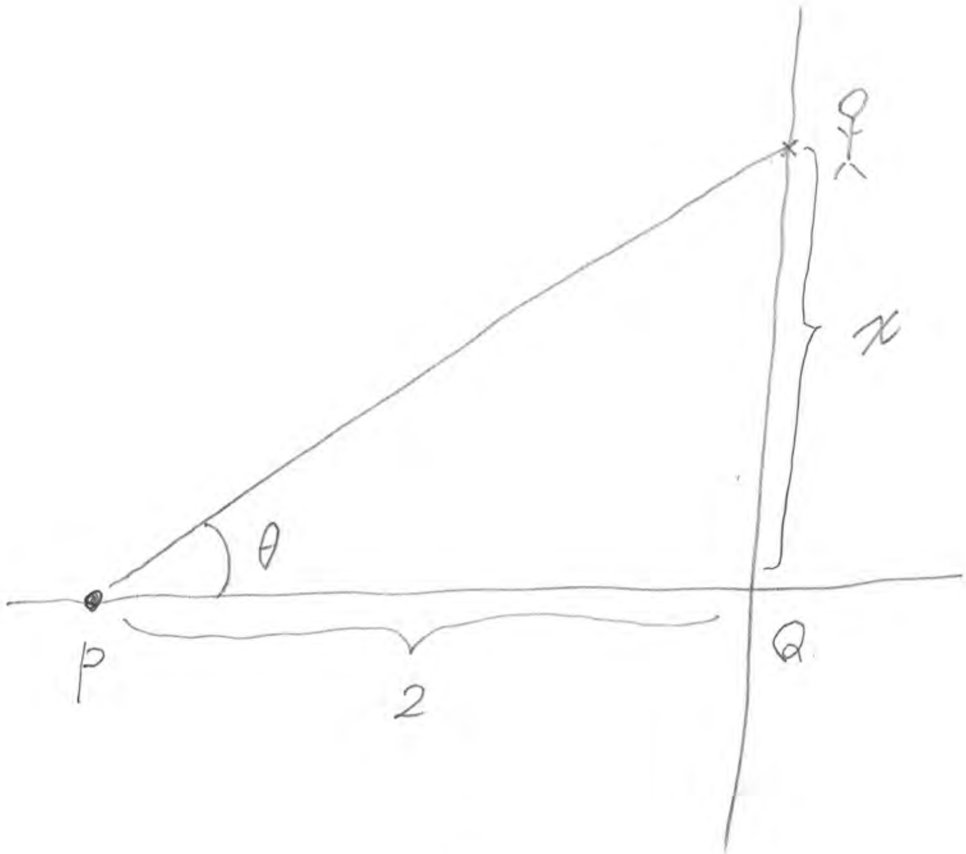

$$\tan\theta = \frac{x}{2}$$

$$\longrightarrow \quad \theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\longrightarrow$$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{2}{4+x^2}$$

1.6

$$y = f(x) = x^3 + 6x + 5$$

$$f'(x) = 3x^2 + 6$$

$(2, 25)$ is on the graph of $y = f(x)$

$\rightarrow$ $(25, 2)$ is on the graph of $y = f^{-1}(x)$

$$(f^{-1})'(25) = \frac{1}{f'(2)} = \frac{1}{18}.$$

eq. of tan. to $y = f^{-1}(x)$:

$$y - 2 = \frac{1}{18}(x - 25)$$

2.1.

(i) $y = x^x$

$$\ln y = \ln(x^x)$$

$$= x \ln x$$

$$\frac{d}{dx}(\quad) = \frac{d}{dx}(\quad)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \cdot \{\quad\}$$

$$= x^x \{\ln x + 1\}$$

(ii) $y = (\ln x)^{\tan 3x}$

$$\ln y = \tan 3x \cdot \ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sec^2(3x) \cdot 3 \cdot \ln(\ln x) + \tan(3x) \cdot \frac{1/x}{\ln x}$$

$$\frac{dy}{dx} = (\ln x)^{\tan(3x)} \left\{ 3\sec^2(3x) \cdot \ln(\ln x) + \tan(3x)\frac{1}{x \ln x} \right\}$$

(iii) $$y = (\sqrt{x})^{\sin x}$$

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$$\ln y = \sin x \cdot \ln\sqrt{x}$$
$$= \frac{1}{2}\sin x \cdot \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = (\sqrt{x})^{\sin x} \cdot \frac{1}{2}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)$$

(iv) $$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \left(-\frac{1}{x^2}\right)\ln x + \frac{1}{x}\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = x^{\frac{1}{x}} \cdot \frac{1}{x^2}\left(1 - \ln x\right)$$
$$= x^{\frac{1}{x}-2}\left(1 - \ln x\right)$$

3.1.

19

$$f(x) + x^2 \cdot (f(x))^3 = 10.$$

$$f'(x) + 2x \cdot (f(x))^3$$
$$+ x^2 \cdot 3(f(x))^2 \cdot f'(x) = 0.$$

$$f'(1) + 2 \cdot 1 \cdot (f(1))^3$$
$$+ 1^2 \cdot 3(f(1))^2 \cdot f'(1) = 0$$

$$f'(1) + 16 + 12 \cdot f'(1) = 0.$$

$$13 f'(1) = -16$$

$$f'(1) = -\frac{16}{13}$$

3.2

20

$$x^2 + 2xy - y^2 + x = 2$$

$$2x + 2 \cdot y + 2x \cdot \frac{dy}{dx} - 2y \frac{dy}{dx} + 1 = 0$$

$$(2x - 2y)\frac{dy}{dx} = -1 - 2x - 2y$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 2y}$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-1 - 2 \cdot 1 - 2 \cdot 2}{2 \cdot 1 - 2 \cdot 2}$$

$$= \frac{-7}{-2} = \frac{7}{2}$$

3.3

21

$$y^2 \cdot \ln x + y = 3x$$

$$2y \cdot \frac{dy}{dx} \cdot \ln x + y^2 \cdot \frac{1}{x} + \frac{dy}{dx} = 3$$

$$(2y \cdot \ln x + 1) \frac{dy}{dx} = 3 - \frac{y^2}{x}$$

$$\frac{dy}{dx} = \frac{3 - \frac{y^2}{x}}{2y \cdot \ln x + 1}$$

$$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{3 - \frac{3^2}{1}}{2 \cdot 3 \cdot \ln 1 + 1} = -6$$

Eq. of Tan.

$$y - 3 = (-6)(x - 1)$$

3.4.

22

$$\ln(x^2 - 3y) = x - y - 1.$$

$$\frac{2x - 3\frac{dy}{dx}}{x^2 - 3y} = 1 - \frac{dy}{dx}$$

$$2x - 3 \cdot \frac{dy}{dx} = (x^2 - 3y)\left(1 - \frac{dy}{dx}\right)$$

$$(x^2 - 3y - 3)\frac{dy}{dx} = x^2 - 3y - 2x$$

$$\frac{dy}{dx} = \frac{x^2 - 3y - 2x}{x^2 - 3y - 3}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{2^2 - 3 \cdot 1 - 2 \cdot 2}{2^2 - 3 \cdot 1 - 3} = \frac{-3}{-2}$$

$$= \frac{3}{2}.$$

Eq. of tan.

$$y - 1 = \frac{3}{2}(x - 2)$$

3.5

$$e^{\frac{x}{y}} = 7x - y$$

$$e^{\frac{x}{y}} \cdot \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2} = 7 - \frac{dy}{dx}$$

$$\left(1 - \frac{x}{y^2} e^{\frac{x}{y}}\right) \frac{dy}{dx} = 7 - \frac{1}{y} e^{\frac{x}{y}}$$

$$\frac{dy}{dx} = \frac{7 - \frac{1}{y} e^{\frac{x}{y}}}{1 - \frac{x}{y^2} e^{\frac{x}{y}}}$$

$$= \frac{7y^2 - y e^{\frac{x}{y}}}{y^2 - x e^{\frac{x}{y}}}$$

4.1. $y = b^{-x}$ $\quad$ $y = b^{x}$

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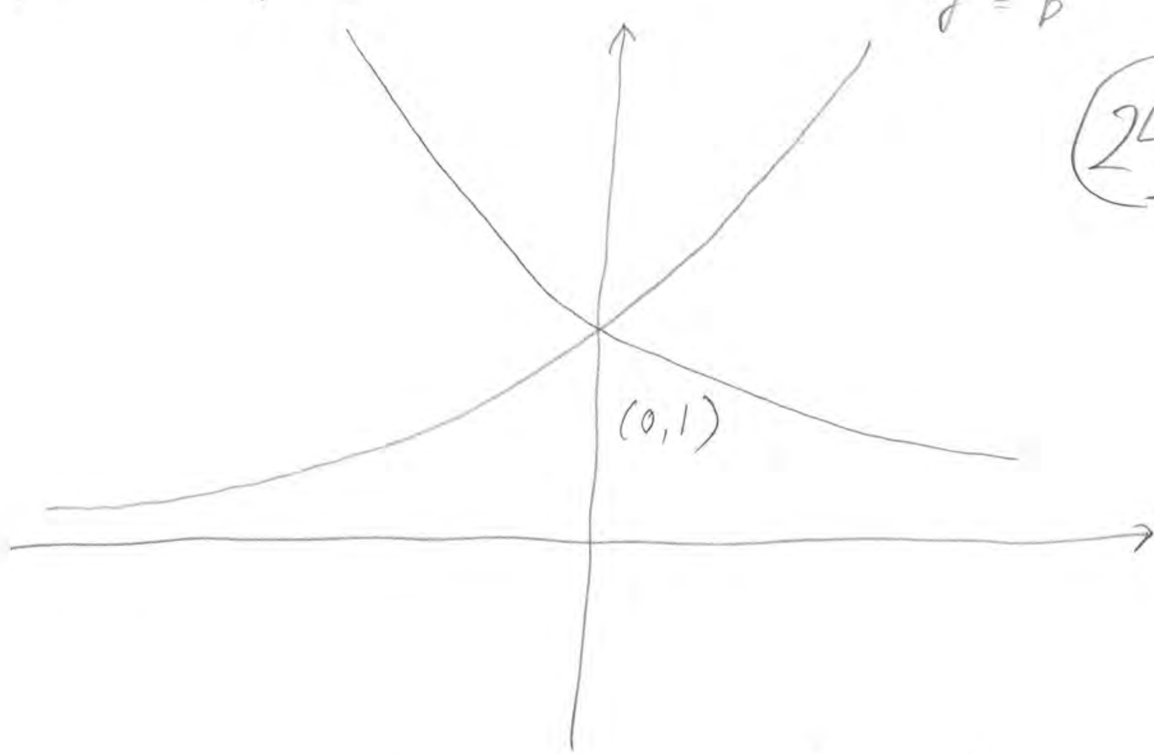


The graphs intersect (at (0,1)!) perpendicularly

$\Longleftrightarrow$ Their tangents at (0,1) intersect perpendicularly

$\Longleftrightarrow$ slope of tan. to $y = b^{x}$ at (0,1)

$\times$ slope of tan. to $y = b^{-x}$ at (0,1)

$= -1$.

Observe

$$y = b^{x}$$

$$\rightarrow y' = b^{x} \cdot \ln b$$

$$y = b^{-x}$$

$$y' = b^{-x} \cdot \ln b \, (-1)$$

$$\Leftrightarrow \quad b^{0} \cdot \ln b \times b^{-0} \cdot \ln b \cdot (-1) = -1$$

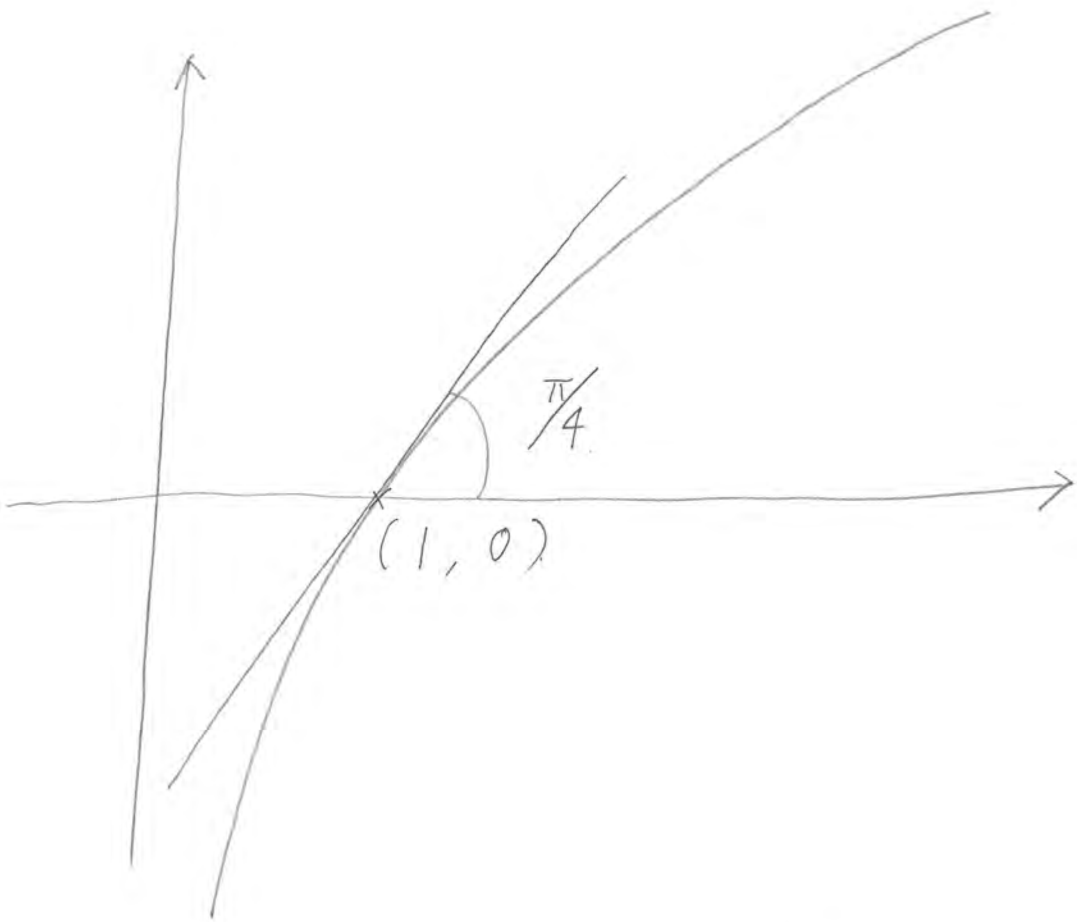
$$\Leftrightarrow \quad (\ln b)^{2} = 1$$

$$\Leftrightarrow \quad \ln b = \pm 1$$

$$\therefore \quad b = e^{\pm 1} = e, \frac{1}{e}$$

4.2.

When $b > 1$, the graph of $y = \log_b x$ takes the shape as below



the graph intersects with the x-axis at an angle of $\pi/4$ at $(1, 0)$

$\Longleftrightarrow$ the tangent to the graph intersects with the x-axis at an angle of $\pi/4$.

$\Longleftrightarrow$ the slope of tan. at $(1, 0) = \tan\frac{\pi}{4}$

$\Longleftrightarrow$

$$y'\Big|_{(1,0)} = (\log_b x)'\Big|_{(1,0)}$$

$$= \left(\frac{1}{\ln b}\cdot\frac{1}{x}\right)\Big|_{(1,0)}$$

$$= \frac{1}{\ln b} = \tan\frac{\pi}{4} = 1$$

i.e.

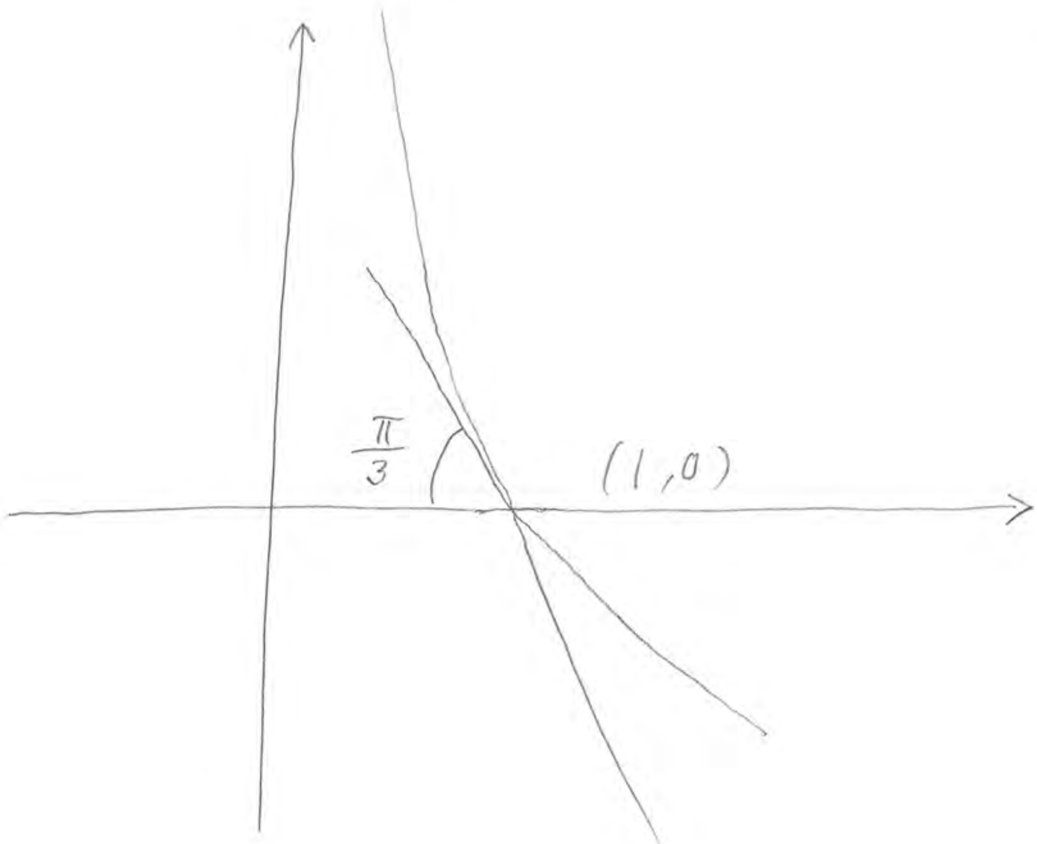
$$\frac{1}{\ln b} = 1$$

$$\longrightarrow \quad \ln b = 1$$

$$\longrightarrow \quad b = e^1 = e$$

4.3. $b\ (<1)$

(28)



The graph of $y = \log_b x$ intersects with the axis at an angle of $\frac{\pi}{3}$

$\iff$ The tangent to the graph of $y = \log_b x$ at $(1, 0)$ intersects with the x-axis at an angle of $\frac{\pi}{3}$.

$\iff$ $$\left.\frac{dy}{dx}\right|_{(1,0)} = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}.$$

$\Leftarrow$

(29)

$$\left.\frac{d}{dx}\left(\log_b x\right)\right|_{(1,0)} = -\sqrt{3}$$

$$\|$$

$$\left.\frac{1}{x\cdot \ln b}\right|_{(1,0)}$$

$$\|$$

$$\frac{1}{\ln b}$$

i.e.

$$\ln b = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$b = e^{-\frac{\sqrt{3}}{3}}$$

5.1

30

(i) $y = \ln\left(x\sqrt{x^2-10}\right)$

$= \ln x + \frac{1}{2}\ln\left(x^2-10\right)$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2-10}$$

$$= \frac{2x^2-10}{x\left(x^2-10\right)}$$

(ii) $y = \ln\left(e^x + xe^x\right)$

$= \ln e^x(1+x)$

$= \ln(e^x) + \ln(1+x)$

$= x + \ln(1+x)$

$$\frac{dy}{dx} = 1 + \frac{1}{1+x} = \frac{2+x}{1+x}$$

(iii) $$y = \frac{(x^3-1)^4 e^x}{(x^2+4)^3}$$

$$\ln y = \ln \left(\qquad \right)$$

$$= 4 \ln |x^3-1| + \underbrace{\ln(e^x)}_{=x}$$
$$- 3 \ln (x^2+4)$$

$$\frac{1}{y}\frac{dy}{dx} = 4 \cdot \frac{3x^2}{x^3-1} + 1$$
$$- 3 \cdot \frac{2x}{x^2+4}$$

$$\frac{dy}{dx} = y \cdot \{ \qquad \}$$

$$= \frac{(x^3-1)^4 e^x}{(x^2+4)^3}$$

$$\times \left\{ \frac{12x^2}{x^3-1} + 1 - \frac{6x}{x^2+4} \right\}.$$

6.1

(i)

$$\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{2}+5h\right)\right]^{7h}-1}{h}$$

Consider

$$\begin{cases} f(x) = \sin\left(\frac{\pi}{2}+5x\right)^{7x} \\ a = 0 \end{cases}$$

Then

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= f'(a)$$

Now we compute

$f'(x)$ when $f(x) = \sin\left(\frac{\pi}{2}+5x\right)^{7x}$

Set $y = \sin\left(\frac{\pi}{2} + 5x\right)^{7x}$ (33)

$$\ln y = 7x \cdot \ln\left[\sin\left(\frac{\pi}{2} + 5x\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 7 \cdot \ln\left[\sin\left(\frac{\pi}{2} + 5x\right)\right] + 7x \cdot \frac{\cos\left(\frac{\pi}{2} + 5x\right)}{\sin\left(\frac{\pi}{2} + 5x\right)} \cdot 5$$

$$\frac{dy}{dx} = y \cdot \{ \quad \downarrow \quad \}$$

$$f'(a) = f'(0)$$

$$= \left.\frac{dy}{dx}\right|_{x=0}$$

$$= y\Big|_{x=0} \cdot \{ \quad \}_{x=0.}$$

$$= 1 \cdot \{7 \cdot \ln 1 + 7 \cdot 0 \frac{0}{1} \cdot 5\}$$

$$= 1 \cdot 0 = \boxed{0}$$

(ii) $$\lim_{h \to 0} \frac{(3+2h)^{5+3h} - 3^5}{h}$$ 34

Consider

$$\begin{cases} f(x) = (3+2x)^{5+3x} \\ a = 0. \end{cases}$$

Then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= f'(a)$$

Now we compute

$f'(x)$ when $f(x) = (3+2x)^{5+3x}$

Set

$$y = (3+2x)^{5+3x}$$

$$\ln y = (5+3x) \ln (3+2x)$$

$$\frac{1}{y}\frac{dy}{dx} = 3 \cdot \ln(3+2x) + (5+3x)\frac{2}{3+2x}$$

$$\frac{dy}{dx} = y \cdot \{ \quad \downarrow \quad \}$$

$$f'(a) = f'(0)$$

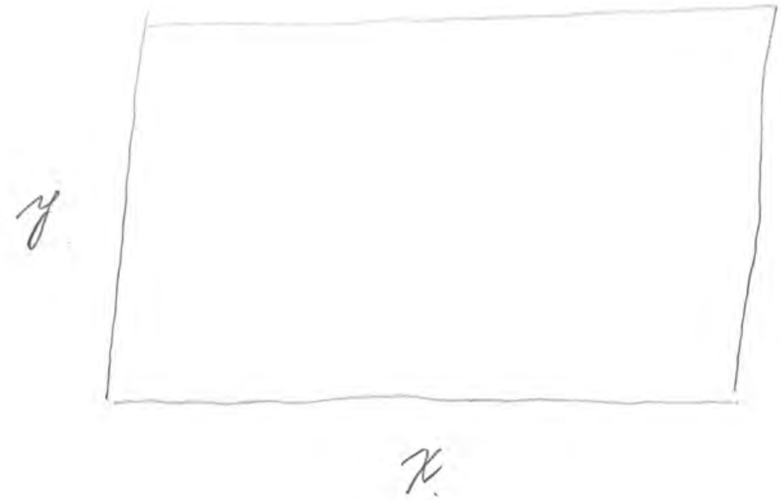
$$= \left.\frac{dy}{dx}\right|_{x=0}$$

$$= y\Big|_{x=0} \cdot \{ \quad \}\Big|_{x=0}$$

$$= 3^5 \cdot \{ 3 \cdot \ln 3 + 5 \cdot \frac{2}{3} \}$$

$$= \boxed{3^5 \left(3 \ln 3 + \frac{10}{3} \right)}$$

7.1.



Given $\frac{dx}{dt} = \frac{dy}{dt} = 2$.

Unknown

$$\frac{dA}{dt} = ? \quad \text{when } t = 3.$$

Relation

$$A = xy.$$

Solution

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= 2 \cdot (7 + 2 \cdot 3) + (3 + 2 \cdot 3) \cdot 2$$

$$= \underbrace{2 \cdot 13}_{26} + \underbrace{9 \cdot 2}_{18} = 44$$

7.2.

Given $\dfrac{dl}{dt} = 5 \qquad \dfrac{dw}{dt} = 4$

Unknown

$$\frac{dA}{dt} = ? \quad \text{when } l = 30, \; w = 25$$

Relation

$$A = lw$$

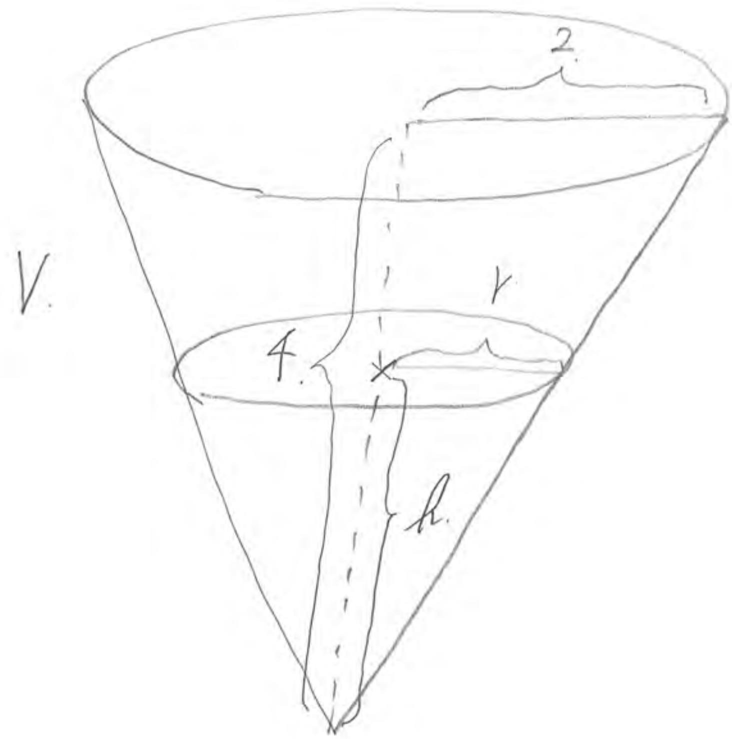
Solution

$$\frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}$$

$$= 5 \cdot 25 + 30 \cdot 4$$

125 120

$$= 245$$



Given $\dfrac{dV}{dt} = 2$

Unknown

$$\frac{dh}{dt} = ? \quad \text{when } h = 3$$

Relation

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{2}{4}h\right)^2 \cdot h$$

$$= \frac{\pi}{12} h^3$$

Solution

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt}$$

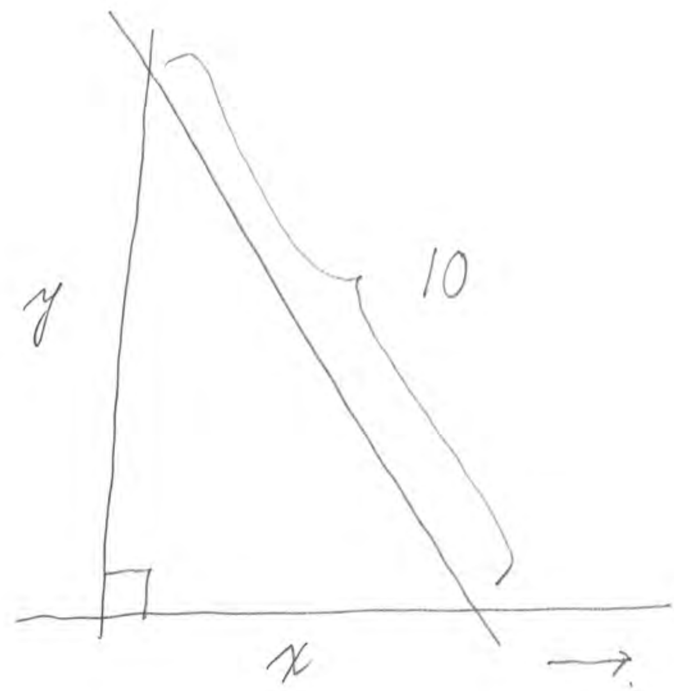
$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{\pi}{12} \cdot 3h^2}$$

$$= \frac{2}{\frac{\pi}{\cancel{12}_{4}} \cdot \cancel{3} \cdot 3^2}$$

$$= \frac{8}{9\pi}$$

10.4.

40



Given $\dfrac{dx}{dt} = 1.$

Unknown

$$\frac{dy}{dt} = ? \quad \text{when} \quad x = 6$$

Relation

$$x^2 + y^2 = 10^2$$

Solution $\sqrt{10^2 - 6^2} = 8$

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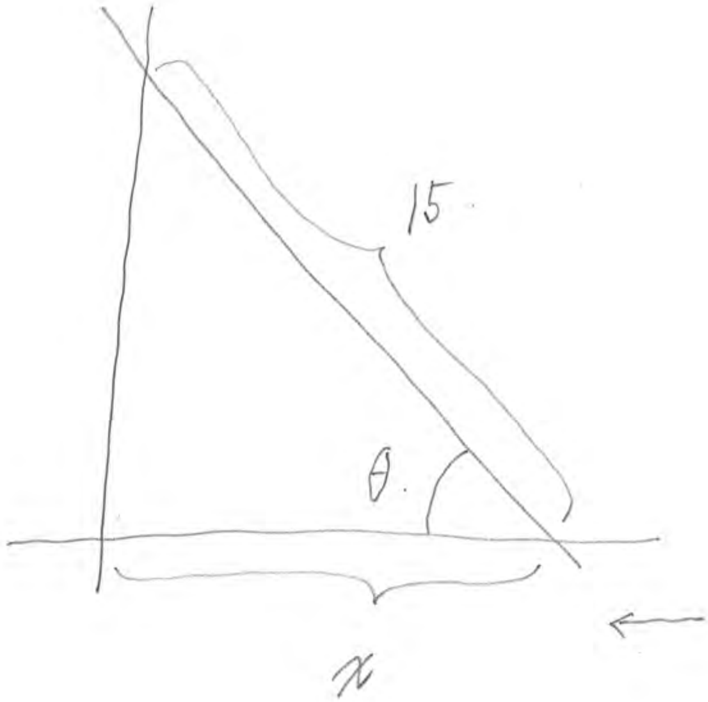
$$2\underset{\substack{\| \\ 6}}{x} \cdot \underset{\substack{\| \\ 1}}{\frac{dx}{dt}} + 2\overset{\substack{8 \\ \|}}{y} \cdot \underset{\substack{\| \\ ?}}{\frac{dy}{dt}} = 0.$$

$$\frac{dy}{dt} = -\frac{2 \cdot 6 \cdot 1}{2 \cdot 8} = -\frac{3}{4}.$$

Ans. The top of the ladder is sliding down at the speed of $+\frac{3}{4}$.

10.5



Given $\dfrac{dx}{dt} = -2$

Unknown.

$$\frac{d\theta}{dt} = ? \quad \text{when} \quad \theta = \frac{\pi}{4}$$

Relation

$$\frac{x}{15} = \cos\theta$$

i.e.

$$x = 15\cos\theta.$$

Solution

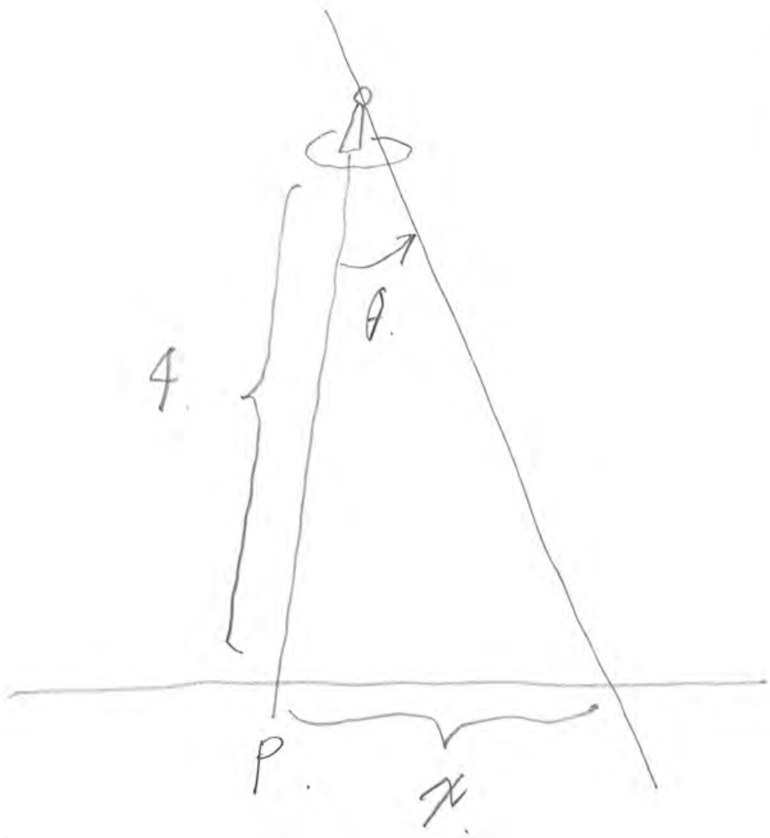
$$\frac{dx}{dt} = 15\underbrace{(-\sin\theta)}_{= -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}}\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = -2$$

$$\frac{d\theta}{dt} = \boxed{\frac{2\sqrt{2}}{15}}$$

10.6

44



Given

$$\frac{d\theta}{dt} = 2\pi \cdot 5 = 10\pi$$

Unknown

$$\frac{dx}{dt} = ? \quad \text{when } x = 1.$$

Relation

$$\frac{x}{4} = \tan\theta.$$

i.e.

$$x = 4\tan\theta$$

Solution.

(45)

$$\frac{dx}{dt} = 4 \cdot \sec^2\theta \cdot \frac{d\theta}{dt}$$

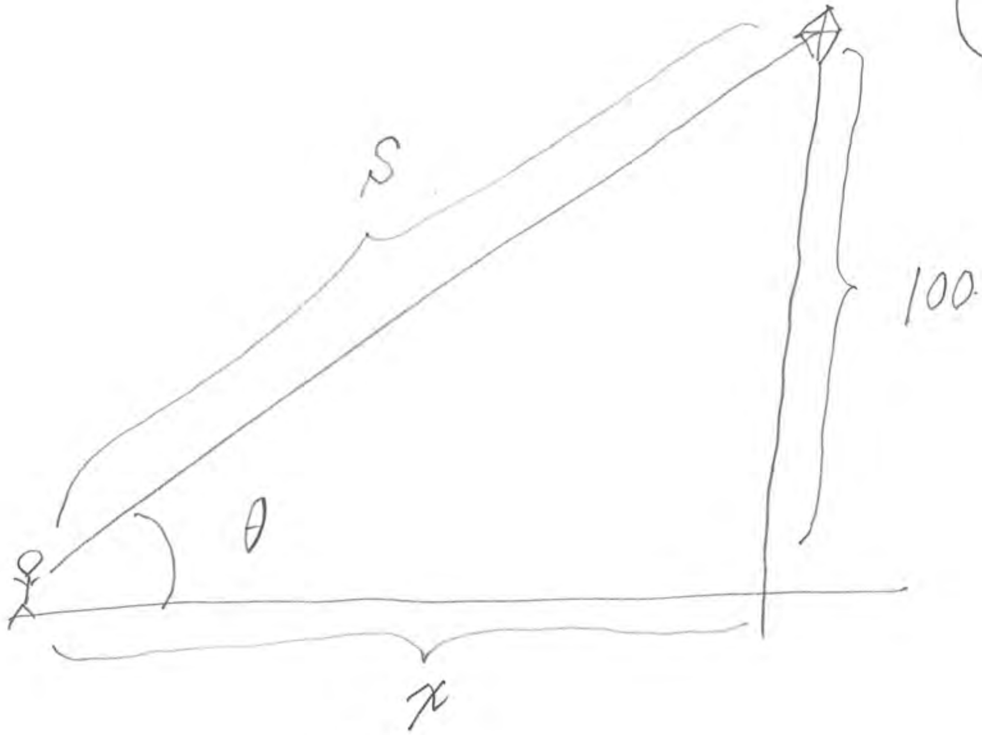
$$\left(\sec\theta = \frac{\sqrt{4^2+1^2}}{4} = \frac{\sqrt{17}}{4} \right)$$

$$= 4 \cdot \left(\frac{\sqrt{17}}{4}\right)^2 \cdot 10\pi$$

$$= \frac{170}{4}\pi$$

10.7.

46



Given $\frac{dx}{dt} = 3$

Unknown

$\frac{d\theta}{dt} = ?$ when $s = 200$.

Relation

$\frac{100}{x} = \tan\theta$.

i.e.

$100 = x \cdot \tan\theta$

Solution.

$$0 = \overset{=3}{\frac{dx}{dt}} \cdot \overset{= \frac{100}{\sqrt{200^2-100^2}} = \frac{1}{\sqrt{3}}}{\tan\theta}$$

$$+ \underset{= 100\sqrt{3}}{x} \cdot \overset{= \left(\frac{2}{\sqrt{3}}\right)^2}{\sec^2\theta} \cdot \frac{d\theta}{dt}$$

$$\sec\theta = \frac{200}{\sqrt{200^2-100^2}} = \frac{2}{\sqrt{3}}$$

i.e.

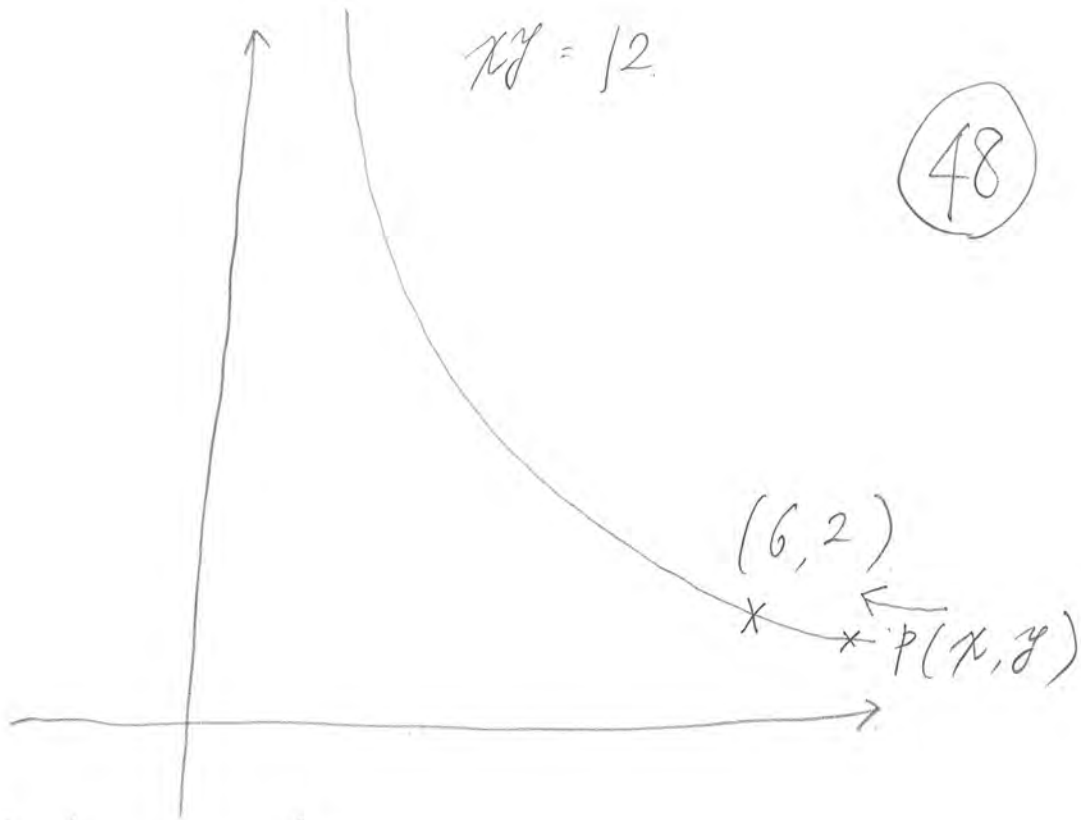
$$0 = 3 \cdot \frac{1}{\sqrt{3}} + 100 \cdot \sqrt{3} \cdot \frac{4}{3} \cdot \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{-3 \cdot \frac{1}{\sqrt{3}}}{100 \cdot \sqrt{3} \cdot \frac{4}{3}} = -\frac{3}{400}$$

Ans. The angle is decreasing at the rate of

$$\frac{3}{400} \text{ rad/sec.}$$

10.8



Given $\frac{dx}{dt} = -5$

Unknown

$$\frac{dy}{dt} = \,? \quad \text{when } (x, y) = (6, 2)$$

Relation

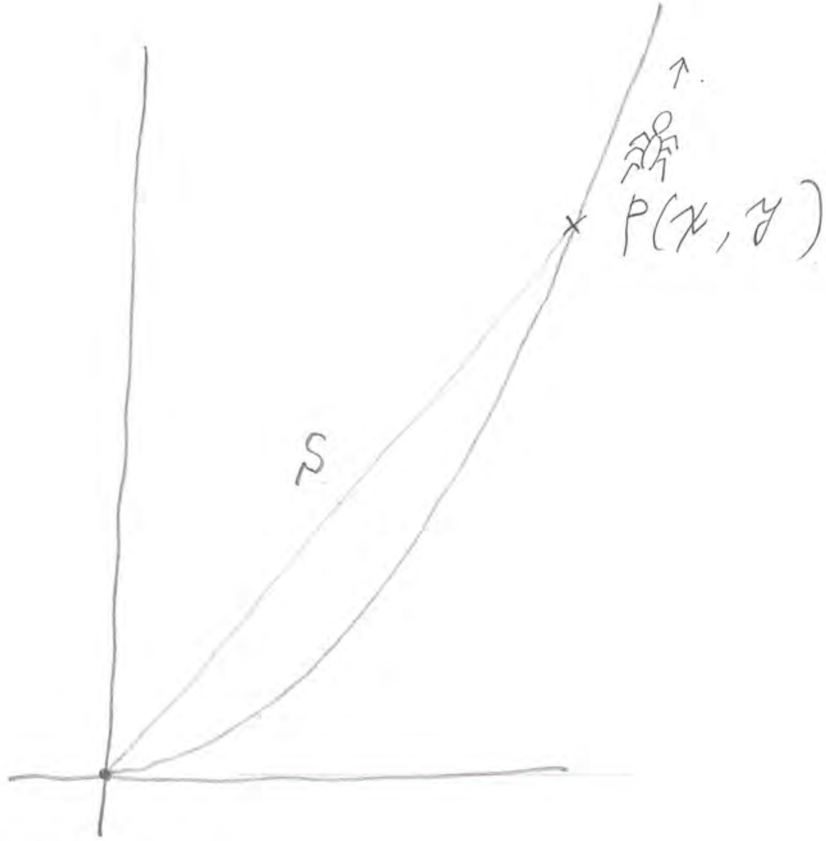
$$xy = 12$$

Solution

$$\overset{=-5}{\frac{dx}{dt}} \cdot \overset{=2}{y} + \overset{=6}{x} \cdot \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

10.9.



(49)

Given

$$\frac{ds}{dt} = 1$$

Unknown

$$\frac{dx}{dt} = ? \quad \text{when } (x, y) = (2, 4)$$

Relation

$$s = \sqrt{x^2 + y^2}$$

$$s^2 = x^2 + y^2 = x^2 + (x^2)^2$$

$$= x^2 + x^4$$

Solution

$$2\underset{\substack{\| \\ \sqrt{2^2+4^2} \\ \| \\ \sqrt{20} \\ \| \\ 2\sqrt{5}}}{S}\;\overset{=1}{\frac{dS}{dt}} = 2\underset{\substack{\| \\ 2}}{x}\cdot\frac{dx}{dt} + 4\underset{\substack{\| \\ 2}}{x}^{3}\cdot\frac{dx}{dt}$$

i.e.

$$2\cdot 2\sqrt{5}\cdot 1 = \overset{4}{2\cdot 2}\,\frac{dx}{dt} + \overset{32}{4\cdot 2^3}\,\frac{dx}{dt}$$

$$4\sqrt{5} = 36\,\frac{dx}{dx}$$

$$\frac{dx}{dt} = \frac{4\sqrt{5}}{36} = \boxed{\frac{\sqrt{5}}{9}}$$

10.10 Purdue Clock Tower

9:00 pm case.

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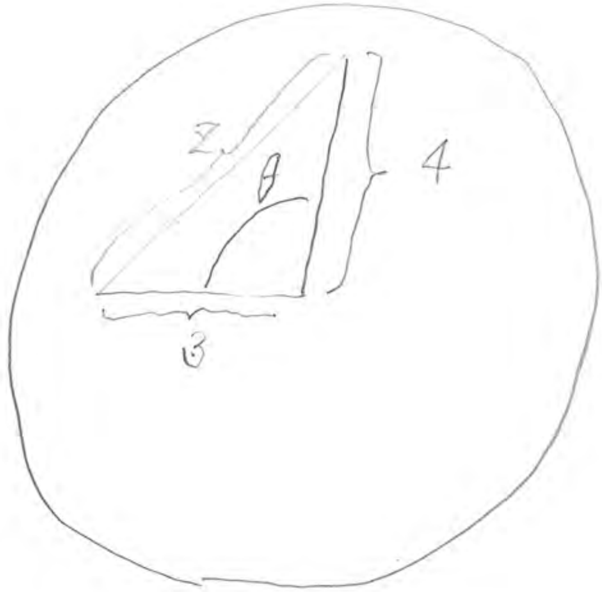


Given

$$\frac{d\theta}{dt} = \frac{2\pi}{60} - \frac{2\pi}{12\cdot 60} = \frac{11\pi}{360}$$

Unknown

$$\frac{dz}{dt} = ? \quad \text{at } 9:00\text{ pm}$$

Relation

$$z^2 = 4^2 + 3^2 - 2\cdot 4\cdot 3\cos\theta$$

Solution

$$2\underbrace{z}_{=\sqrt{4^2+3^2}}\frac{dz}{dt} = -2\cdot 12\cdot\Big(-\sin\overbrace{\theta}^{=\frac{\pi}{2}}\Big)\cdot\underbrace{\frac{d\theta}{dt}}_{=\frac{11\pi}{360}}$$

i.e.

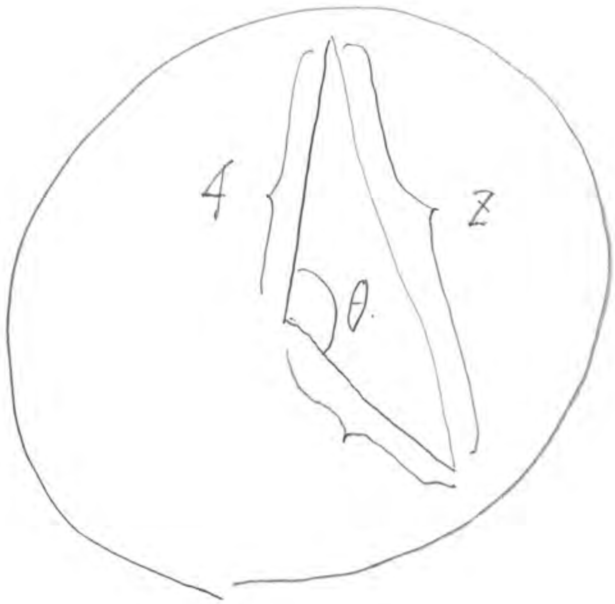
$$2 \cdot 5 \cdot \frac{dz}{dt} = -2 \cdot 12 \cdot (-1) \cdot \frac{11\pi}{360}$$

$$\frac{dz}{dt} = \frac{\cancel{12}}{5} \cdot \frac{11\pi}{\underset{30}{\cancel{360}}} = \frac{11\pi}{150}.$$

4:00 pm case

is left as an exercise

for you.



4:00 pm
case

53

Given

$$\frac{d\theta}{dt} = \frac{2\pi}{12\cdot 60} - \frac{2\pi}{60} = -\frac{11\pi}{360}.$$

Unknown

$$\frac{dz}{dt} = ? \quad \text{at} \quad 4:00\ pm$$

Relation

$$z^2 = 4^2 + 3^2 - 2\cdot 4\cdot 3 \cos\theta$$

Solution

$$2z\frac{dz}{dt} = -2\cdot 12\,(-\underbrace{\sin\overset{\frac{2\pi}{3}}{\theta}}_{\frac{\sqrt{3}}{2}})\cdot \overset{-\frac{11\pi}{360}}{\frac{d\theta}{dt}}$$

$$z^2 = 4^2 + 3^2 - 2 \cdot 4 \cdot 3 \underbrace{\cos\left(\frac{2\pi}{3}\right)}_{-\frac{1}{2}}$$

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$$= \underset{16}{4^2} + \underset{9}{3^2} + \underset{12}{4 \cdot 3} = 37$$

$$z = \sqrt{37}$$

$$\cancel{2} \cdot \sqrt{37} \cdot \frac{dz}{dt} = -\cancel{2} \cdot \cancel{12} \cdot \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{11\pi}{\cancel{360}_{30}}\right)$$

$$\frac{dz}{dt} = -\frac{11\sqrt{3} \cdot \pi}{60\sqrt{37}}$$