Study Guide for Exam 1

1. You are supposed to know and understand the basics about the exponential function, and about the logarithmic function as the inverse of the exponentai function. You should be able to solve the equations involving the exponential and logarithmin functions.

Example Problems

1.1. Solve the following equations.
(i) $\ln x + \ln(x-1) = 0$
(ii) $\log_5 x^2 + 2\log_5 x = \log_5 81$
(iii) $\log_2 5 \cdot \ln 2 = \ln x$
(iv) $e^{x^2-3x+2} = 1$

2. Having the information on the range of a given rotation angle θ and knowing the value of a trigonometric function, you are supposed to be able to determine the values of the other trigonometric functions.

Example Problems

2.1. We have the information

$$\sin\theta = \frac{3}{5} \text{ and } \frac{\pi}{2} < \theta < \pi.$$

Determine the values of $\cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta$.

2.2. We have the information

$$\tan\theta = -\frac{5}{12}, \quad \sin\theta > 0.$$

Determine the values of $\sin\theta, \cos\theta, \csc\theta, \sec\theta, \cot\theta$.

3. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formula for sine and cosine, $\sin^2 x + \cos^2 x = 1$, etc.).

Example Problems

3.1. Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$\cos x = \cos(2x).$$

3.2. Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$3\cot x = 2\sin(2x).$$

3.3. Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$\tan^2 x - 3 = 0.$$

4. You are supposed to know the domain and range of each of the inverse trigonometric functions $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$. You are also supposed to know the basic identities involving the inverse trigonometric functions.

Example Problems

4.1. Find the exact values for the following

(i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(ii) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(iii) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

(iv) $\sin^{-1}\left(\sin\left(\frac{8\pi}{7}\right)\right)$

(v) $\cos^{-1}\left(\sin\left(\frac{9\pi}{8}\right)\right)$

(vi) $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)$

(vii) $\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(-\frac{1}{3}\right)$

(viii) $\cos^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(-\frac{1}{5}\right)$

(ix) $\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right) + \tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right)$

5. You should know the condition (one-to-one) for a function to have its inverse. Given a function which is one-to-one, you are supposed to be able to find the formula of the inverse function, its domain and range, and draw the graph of the inverse function.

Example Problems

5.1. Find the formula, and state the domain and range of the inverse of the function

$$y = \frac{1 - e^{-x}}{1 + e^{-x}}.$$

5.2.

(i) Consider the function $y = f(x) = x^2 - 1$ over the interval $(-2, 0]$. Find the formula, and state the domain and range of its inverse.

(ii) Consider the function $y = f(x) = x^2 - 1$ over the interval $[1, 3)$. Find the formula, and state the domain and range of its inverse.

5.3. Consider the function $y = \dfrac{3x+1}{-5x+3}$. Find the formula representing the function whose graph is symmetric, with respect to the line $y = x$, to the graph of the given function.

5.4. Consider the function $f(x) = \left(e^{|x-5|}\right) 2 - 1$ over the domain $[1, 3]$. Does the inverse of the function exist ? If it does, then write down the formula for the inverse function $f^{-1}(x)$, its doamin and range.

6. You are supposed to be able to compute the (right/left hand side) limit, understanding its proper meaning, and using the Squeeze Theorem. You are also supposed to be able to determine the exact value of the limit who has an indeterminate form (e.g. $0/0, \pm\infty/\pm\infty, \infty-\infty$) using some proper technique.

Example Problems

6.1. Compute the following limits:

(i) $\lim_{x\to 5^-} \dfrac{x^2-3x-5}{|x-5|}$

(ii) $\lim_{x\to(\pi/2)^-} \tan x$

(iii) $\lim_{x\to(-\pi)^-} \cot x$

(iv) $\lim_{x\to(\pi/2)^+} e^{\tan x}$

(v) $\lim_{x\to 2^-} \left(\dfrac{x^2+5x-14}{x^2-6x+8}\right)$

(vi) $\lim_{x\to 0} \left(\dfrac{5}{x^2-x} + \dfrac{5}{x}\right)$

(vii) $\lim_{x\to\infty} \left(\sqrt{x^2+3x+1} - x\right)$

(viii) $\lim_{x\to 0} \dfrac{|3x-4| - |5x+4|}{x}$

(ix) $\lim_{x\to 0} \dfrac{\sin(1/x)}{1/x}$

(x) $\lim_{x\to 8} \dfrac{\sqrt[3]{x}-2}{x-8}$

(xi) $\lim_{x\to\infty} \dfrac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$

(xii) $\lim_{x\to-\infty} \dfrac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$

7. You are supposed to understand the meaning of the Intermediate Value Theorem, and to be able use it to show that a certain equation has a root in a specified interval.

Example Problems

7.1. Using the I.V. Th., determine on which of the following intervals

$$(0,1),(1,2),(2,3),(3,4)$$

the equation $x^3 - 3x = 5$ has a root.

7.2. Prove that the equation $\sin x = x^3 + 1$ has at least one real root.

8. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

Example Problems

8.1. Suppose we have a function $f(x)$ with $f'(2) = 5$.
Determine the following values:
(i) $g'(1)$ where $g(x) = f(2x)$.
(ii) $\lim_{h\to 0} \dfrac{f(2+4h) - f(2)}{3h}$.
(iii) $\lim_{h\to 0} \dfrac{f(2+4h) - f(2-5h)}{7h}$.

8.2. Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} -(x-1)^2 + 3 & \text{if} \quad x < 0 \\ 5x + 2 & \text{if} \quad x \geq 0. \end{cases}.$$

Does $f'(0)$ exist ? If it exists, compute the value of $f'(0)$.

9. When a function is defined piecewise and depending on some variables, you are supposed to know how to determine those variables so that the function becomes continuous entirely over the specified interval (e.g. everywhere).

Example Problems

9.1. Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 - a & \text{if} \quad x \leq 1 \\ \dfrac{3x^2 + 12x - b}{x^2 + 2x - 3} & \text{if} \quad x > 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

9.2. Consider the following function

$$f(x) = \begin{cases} 3x - 2c & \text{if } x \le c \\ 5x^2 - 4 & \text{if } x > c. \end{cases}$$

Determine all the values of c so that f is continuous everywhere.

10. You are supposed to be able to find the horizontal/vertical asymptote(s) of a given function.

Example Problems

10.1. Find the horizontal/vertical asymptote(s) of the following functions

(i) $y = f(x) = \dfrac{x^3 + 4x^2 + x - 6}{x(x^2 - 1)}$.

(ii) $y = f(x) = \dfrac{x^2 - x}{x^2 - 4x + 3}$.

(iii) $y = f(x) = \dfrac{3e^x}{e^x - 1}$.

(iv) $y = f(x) = \dfrac{x - 5}{\sqrt{x^2 - 5x}}$.

11. You are supposed to be able to compute the derivative of a polynomial (square root/rational) function, and understand that its value represents the slope of the tangent line to the graph of the function.

Example Problems

11.1. Find the equation of the line that is tangent to the curve $y = x^3$ and is

(i) parallel to the line $y = 3x$,

(ii) perpendicular to the line $y = 3x$.

11.2. Find the equation of the line(s) which is tangent to the parabola $y = x^2$ and passes thorugh the point $(1, -3)$.

12. You are supposed to understand the meaning of the continuity and differentiability, and their difference.

Example Problems

12.1. Determine where the followig function is continuous / differentiable.

(i) $f(x) = \begin{cases} \dfrac{x - 2}{|x - 2|} & \text{if } x \ne 2 \\ 0 & \text{if } x = 2. \end{cases}$

(ii) $g(x) = x|x - 2|$

(iii) $h(x) = x(x - 2)|x - 2|$

12.2. **Challenge Problem**: Consider the function described below:

$$g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Choose the right statement about the continuity and differentiability of the function $y = g(x)$ at 0.

A. The function g is continuous at 0 and differentiable at 0.
B. The function g is continuous at 0 but not differentiable at 0.
C. The function g is not continuous at 0 but differentiable at 0.
D. The function g is not continuous at 0 and not differentiable at 0.
E. The above description is not sufficient to judge the continuity or the differentiability of the function g.