Study Guide for Exam 2

1. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are supposed to understand the relation between the derivative of a one-to-one function and that of its inverse. You are supposed to be able to compute the derivatives of the inverse trigonometric functions.

Example problems:

1.1. Compute the derivative of the following function:

(i) $y = \sin(\sin(\sin x))$
(ii) $y = \cos(2\pi \cdot 3^x)$
(iii) $y = \left(\dfrac{t-2}{2t+1}\right)^9$
(iv) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$ (iv*) $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$
(v) $y = e^{\sec 3\theta}$
(vi) $y = e^{2^{x^3}}$
(vii) $y = \sin^{-1}(1/x)$
(viii) $y = \tan^{-1}(\sqrt{x})$
(ix) $y = \ln|\sec(3\theta) + \tan(3\theta)|$
(x) $y = \ln(e^{\sin x} + e^{-\sin x})$
(xi) $y = \dfrac{e^x}{\sqrt{x^2+1}}$
(xii) $y = \ln(\sin(x^2))$

1.2. Suppose that $F(x) = f^{-1}\left(\{g(x)\}^2\right)$ and that the functions f (which is one-to-one, and hence has its inverse) and g satisfy the following conditions. Find $F'(1)$.

$$\begin{cases} f(2) = 9, \quad f(9) = 5, \\ f'(1) = 4, \quad f'(2) = 3, \quad f'(3) = -2 \\ g(1) = 3, \quad g'(1) = 2 \end{cases}$$

1.3. Suppose that $F(x) = [g(f^{-1}(x))]^2$,

$$f(2) = 9, f'(2) = 5, g(2) = 3, g'(2) = -2.$$

Find the value of $F'(9)$.

1.4. Problem 81 on Page 227 of the textbook.

1.5. A person walks along a straight path, and a sensor is placed at the point P which is 2 meters away from the closest point Q on the path. Let x be the distance between the person and the point Q, and let θ be the angle formed by the line PQ and the line connecting the sensor and the person. Compute $d\theta/dx$.

1.6. Consider the function $y = f(x) = x^3 + 6x + 5$. It is one-to-one, and hence has its inverse function f^{-1}.

Observe that the point $(2, 25)$ is on the graph of $y = f(x)$.

Find the equation of the tangent line to the graph of $y = f^{-1}(x)$ at the point $(25, 2)$.

2. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems:

2.1. Find the derivative of the following function.
(i) $y = x^x$
(ii) $y = (\ln x)^{\tan 3x}$
(iii) $y = (\sqrt{x})^{\sin x}$
(iv) $y = x^{1/x}$

3. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems:

3.1. Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the following equation

$$f(x) + x^2 (f(x))^3 = 10$$

and that $f(1) = 2$. Find $f'(1)$.

3.2. Find the slope of the tangent to the curve given by the equation

$$x^2 + 2xy - y^2 + x = 2$$

at point $(x, y) = (1, 2)$

3.3. Find the equation of the tangent line to the curve defined by $y^2(\ln x) + y = 3x$ at the point $(1, 3)$.

3.4. Find the equation of the tangent line to the curve defined by

$$\ln(x^2 - 3y) = x - y - 1$$

at the point $(2, 1)$

3.5. Find $\dfrac{dy}{dx}$ given $e^{x/y} = 7x - y$.

4. You are supposed to be able to compute the derivatives of the exponential functions and logarithmic functions, as well as to understand the shape of the graphs of these functions.

Example Problems:

4.1. Find all such values of $b > 0$ that the graph of $y = b^x$ and the graph of $y = b^{-x}$ intersect perpendicularly.

4.2. Find the value b (> 1) such that the graph of $y = \log_b x$ intersects with the x-axis at an angle of $\pi/4$.

4.3. Find the value b (< 1) such that the graph of $y = \log_b x$ intersects with the x-axis at an angle of $\pi/3$.

5. You are supposed to be able to compute the derivative of a function involving the logarithmic finctions, first simplifying the formula using the laws of the logarithms.

Example Problems:

5.1. Compute the derivatives of the following functions.

(i) $y = \ln(x\sqrt{x^2 - 10})$

(ii) $y = \ln(e^x + xe^x)$

(iii)$y = \dfrac{(x^3 - 1)^4 e^x}{(x^2 + 4)^3}$

6. You are supposed to be able to compute the limit of some indetrminate form, by relating its computation to the definition of the derivative.

Example Problems:

6.1. Compute the following limits.

(i) $\lim_{h \to 0} \dfrac{\left[\sin\left(\frac{\pi}{2} + 5h\right)\right]^{7h} - 1}{h}$

(ii) $\lim_{h \to 0} \dfrac{(3 + 2h)^{5+3h} - 3^5}{h}$.

7. FOUR "Related Rates" problems will be given in Exam 2. The problems are very similar to the ones given as Examples in the textbook, to the problems in MyLabMath, and to the ones given here in the study guide. Of particular importance are:

- Rectangle problem
- Conical tank problem (Coffee maker problem)
- Ladder problem
- Light house problem
- Kite problem
- Particle moving along the graph of a function problem
- Purdue clock tower problem (**Challenge**)

7.1. A rectangle initially has dimensions 3 cm by 7 cm. All sides begin increasing in length at a rate of 2 cm/sec.

At what rate is the area of the rectangle increasing after 3 sec ?

7.2. The length of a rectangle is increasing at a rate of 5 cm/sec and its width is increasing at a rate of 4 cm/sec.

When the length is 30 cm nd the width is 25 cm, how fast is the area of the rectangle increasing ?

7.3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

7.4. (speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

7.5 (angle) A 15-foot plank of wood is leaning against a vertical wall and its bottom is being pushed toward the wall at the rate of 2 ft/sec.

At what rate is the angle θ between the plank and the ground changing when the acute angle the plank makes with the ground is $\pi/4$?

7.6. A lighthouse is located on a small island 4 km away from the nearest point P on a straight shoreline and its light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

7.7. A kite 100 ft above the ground moves horizontally at a speed of 3 ft/sec. At what rate is the angle (in radians) between the string and the horizontal decreasing when 200 ft of string have been let out ?

7.8. A particle is moving along the curve $xy = 12$. As it reaches the point $(6, 2)$, the x-coordinate is decreasing at a rate of 5 cm/sec. What is the rate of change of the y-coordinate of the particle at that instant ?

Note: The unit for measuring the coordinate length is given by "cm".

7.9. A bug is moving along the right side of the parabola $y = x^2$ at a rate such that its distance from the origin is increasing at 1 cm/min.

At what rate is the x-coordinate of the bug increasing at the point $(2, 4)$?

10.10. The long and short hands of the clock tower at Purdue University are 4 ft long and 3 ft long, respectively. How fast is the distance between the tips of the hands changing at 9:00 PM (resp. 4:00 PM) ?

HINT:

①. Let θ be the angle between the long and short hands.

Then $\dfrac{d\theta}{dt} = \dfrac{2\pi}{60} - \dfrac{2\pi}{12 \cdot 60} = \dfrac{11\pi}{360}$ rad/min.

$$\left(\text{resp. } \frac{d\theta}{dt} = \frac{2\pi}{12 \cdot 60} - \frac{2\pi}{60} = -\frac{11\pi}{360} \text{ rad/min}\right).$$

②. Law of cosines reads

$$z^2 = x^2 + y^2 - 2xy\cos(\theta)$$

for a triangle with sides x and y forming an angle θ and the third side z.