

## Study Guide for Final Exam

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1. You are supposed to know and understand the basics about the exponential function, and about the logarithmic function as the inverse of the exponential function. You should be able to solve the equations involving the exponential and logarithmic functions.

### Example Problems

1.1. Solve the following equations.

(i)  $\ln x + \ln(x - 1) = 0$

(ii)  $\log_5 x^2 + 2 \log_5 x = \log_5 81$

(iii)  $\log_2 5 \cdot \ln 2 = \ln x$

(iv)  $e^{x^2-3x+2} = 1$

2. Having the information on the range of a given rotation angle  $\theta$  and knowing the value of a trigonometric function, you are supposed to be able to determine the values of the other trigonometric functions.

### Example Problems

2.1. We have the information

$$\sin \theta = -\frac{12}{13} \quad \text{and} \quad \pi < \theta < \frac{3\pi}{2}.$$

Determine the value of  $\cot \theta$ .

2.2. We have the information

$$\sin \theta = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{3\pi}{2} < \theta < 2\pi.$$

Determine the value of  $\tan \theta$ .

2.3. We have the information

$$\cos 2\theta = -\frac{1}{2} \quad \text{and} \quad 0 < \theta < \pi.$$

Determine the value of  $\sin \theta$ .

3. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formulas for sine and cosine,  $\sin^2 x + \cos^2 x = 1$ , etc.).

### Example Problems

3.1. How many solutions are there on the interval  $[0, 2\pi]$  for the equation  $\sqrt{3} \sin x = \sin(2x)$  ?

3.2. Find all the values  $x$  on the interval  $[0, 2\pi]$  satisfying the equation  $\cos(2x) - \sin(2x) = 0$ .

3.3. Find all the values  $x$  on the interval  $[0, 2\pi]$  satisfying the equation  $\cos(2x) - \sin x = 0$ .

4. Given a function which is one-to-one, with a given domain and range, you are supposed to be able to find the formula for its inverse function, describing its domain and range. You should also understand the relation between the graph of the original function and the one for its inverse.

### Example Problems

4.1. Find a formula for the inverse of the following function, and describe the domain and range of the inverse function.

$$(i) f(x) = \frac{6x - 1}{2x + 1}, (ii) f(x) = \frac{2e^x - 1}{2e^x + 1}, (iii) f(x) = 1 - \sqrt{x + 1}.$$

5. You are supposed to be able to compute the (right/left hand side) limit, understanding its proper meaning, and using the Squeeze Theorem. You are also supposed to be able to determine the exact value of the limit who has an indeterminate form (e.g.  $0/0$ ,  $\pm\infty/\pm\infty$ ,  $\infty - \infty$ ) using some proper technique.

### Example Problems

5.1. Compute the following limits:

$$(i) \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 7x + 12}$$

$$(ii) \lim_{x \rightarrow 5^-} \frac{x^2 - 4x - 5}{|x - 5|}$$

$$(iii) \lim_{x \rightarrow (\pi/2)^-} \tan x$$

$$(iv) \lim_{x \rightarrow (-\pi)^-} \cot x$$

$$(v) \lim_{x \rightarrow (\pi/2)^+} e^{\tan x}$$

$$(vi) \lim_{x \rightarrow 2^-} \left( \frac{x^2 + 5x - 14}{x^2 - 6x + 8} \right)$$

$$(vii) \lim_{x \rightarrow 0} \left( \frac{5}{x^2 - x} + \frac{5}{x} \right)$$

$$(viii) \lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x + 1} - x)$$

$$(ix) \lim_{x \rightarrow 0} \frac{|3x - 4| - |5x + 4|}{x}$$

$$(x) \lim_{x \rightarrow 0} \frac{\sin(1/x)}{1/x}$$

$$(xi) \lim_{x \rightarrow 8} \frac{\sqrt[3]{x} - 2}{x - 8}$$

$$(xii) \lim_{x \rightarrow \infty} \frac{x^3 + 3x + 2}{2x^3 + \sqrt{9x^6 + 4x + 5}}$$

$$(xiii) \lim_{x \rightarrow -\infty} \frac{x^3 + 3x + 2}{2x^3 + \sqrt{9x^6 + 4x + 5}}$$

6. When a function is defined piecewise and depending on some variables, you are supposed to know how to determine those variables so that the function becomes continuous entirely over its domain.

### Example Problems

6.1. Consider the following function

$$f(x) = \begin{cases} 3x - 2c & \text{if } x \leq c \\ 5x^2 - 4 & \text{if } x > c. \end{cases}$$

Determine all the values of  $c$  so that  $f$  is continuous everywhere.

6.2. Find the values of  $a$  and  $b$  so that the function

$$f(x) = \begin{cases} x^2 - a & \text{if } x \leq 1 \\ \frac{3x^2 + 12x - b}{x^2 + 2x - 3} & \text{if } x > 1 \end{cases}$$

is continuous on  $(-\infty, \infty)$ .

7. You are supposed to understand the meaning of the continuity and differentiability, and their difference.

### Example Problems

7.1. Determine where the following function is continuous / differentiable.

$$(i) f(x) = \begin{cases} \frac{x-2}{|x-2|} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2. \end{cases}$$

$$(ii) g(x) = x|x-2|$$

$$(iii) h(x) = x(x-2)|x-2|$$

7.2. Consider the function described below:

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Choose the right statement about the continuity and differentiability of the function  $y = g(x)$  at 0.

- A. The function  $g$  is continuous at 0 and differentiable at 0.
- B. The function  $g$  is continuous at 0 but not differentiable at 0.
- C. The function  $g$  is not continuous at 0 but differentiable at 0.
- D. The function  $g$  is not continuous at 0 and not differentiable at 0.
- E. The above description is not sufficient to judge the continuity or the differentiability of the function  $g$ .

8. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

### Example Problems

8.1. Suppose we have a function  $f(x)$  with  $f'(2) = 5$ .

Determine the following values:

(i)  $g'(1)$  where  $g(x) = f(2x)$ .

(ii)  $\lim_{h \rightarrow 0} \frac{f(2+4h) - f(2)}{3h}$ .

(iii)  $\lim_{h \rightarrow 0} \frac{f(2+4h) - f(2+5h)}{9h}$ .

8.2. Compute the following limit

(i)  $\lim_{h \rightarrow 0} \frac{(3+7h)^5 - 3^5}{h}$ .

(ii)  $\lim_{h \rightarrow 0} \frac{(3+2h)^{5+3h} - 3^5}{h}$ .

8.3. Consider the following limit

$$L = \lim_{h \rightarrow 0} \frac{(2+5h)^{3+5h} - 2^3}{h}.$$

Consider the function  $f(x) = (2+x)^{3+x}$ . Describe  $L$  in terms of  $f'(0)$ .

8.4. Consider the following limit

$$\lim_{x \rightarrow \pi} \frac{e^{\sin x} - 1}{x - \pi}.$$

We would like to interpret it as the derivative of a function  $f(x)$  at  $x = a$ . Find  $f(x)$  and  $a$ .

9. You are supposed to be able to compute the derivative of a function, and understand that its value represents the slope of the tangent line to the graph of the function.

### Example Problems

9.1. Find the equation of the line that is tangent to the curve  $y = \frac{2}{3}x\sqrt{x}$  and is also parallel to the line  $y = 2x + 3$ .

9.2. Find the equation(s) of the tangent line(s) to the graph of a function  $y = x^2$ , passing through the point  $(1, -3)$ .

10. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are also supposed to know the relation between the derivative of the original function and the derivative of its inverse (when it exists).

**Example problems**

10.1. Compute the derivative of the following function:

(i)  $y = \sin(\sin(\sin x))$

(ii)  $y = \left(\frac{t-2}{2t+1}\right)^9$

(iii)  $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

(iv)  $y = e^{\sec 3\theta}$

10.2. Suppose that  $F(x) = f(x)^2 \cdot f(g(x))$  and that the functions  $f$  and  $g$  satisfy the following conditions. Find  $F'(1)$ .

$$\begin{cases} f(1) = 5, & f(2) = 3, & f(3) = -1 \\ f'(1) = 4, & f'(2) = 3, & f'(3) = -2 \\ g(1) = 3, & g(2) = 2, & g(3) = -1 \\ g'(1) = 2, & g'(2) = 3, & g'(3) = 4 \end{cases}$$

10.3. Suppose that  $F(x) = f^{-1}(\{g(x)\}^2)$  and that the functions  $f$  (which is one-to-one, and hence has its inverse) and  $g$  satisfy the following conditions. Find  $F'(1)$ .

$$\begin{cases} f(2) = 9, & f(9) = 5, \\ f'(1) = 4, & f'(2) = 3, & f'(3) = -2 \\ g(1) = 3, & g'(1) = 2 \end{cases}$$

10.4. We have two everywhere differentiable functions  $h$  and  $g$  such that  $x^3 = g(h(x))$  and that  $h(1) = 5$  &  $h'(1) = 7$ . Determine  $g'(5)$ .

11. You are supposed to know how to compute the derivative of a function of the form  $y = f(x)^{g(x)}$ .

**Example Problems**

11.1. Find the derivative of the following function.

(i)  $y = x^x$

(ii)  $y = (\sqrt{x})^{\sin x}$

(iii)  $y = x^{\ln x}$

(iv)  $y = (\cot x)^{\sin x}$

12. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

**Example Problems**

12.1. Suppose that  $f$  is a differentiable function defined on  $(-\infty, \infty)$  satisfying the equation  $f(x) + x^2(f(x))^3 = 10$  and the condition  $f(1) = 2$ . Find  $f'(1)$ .

12.2. Find the slope of the tangent to the curve given by the equation  $x^2 + 2xy - y^2 + x = 2$  at point  $(x, y) = (1, 2)$

12.3. Find  $\frac{dy}{dx}$  given  $e^{x/y} = 7x - y$ .

12.4. Find the equation of the tangent line to the curve defined by  $\sqrt{y} + \sqrt{x} = 3$  at  $(1, 4)$ .

12.5. Consider the curve defined by the equation  $xe^y - ye^x = 2$ . Compute  $\frac{dy}{dx}$  at the point  $(2, 0)$ .

13. You are supposed to be able to provide an approximation of the value of a function, using the linear approximation.

**Example Problems**

13.1. Find the linear approximation  $L(x)$  of the function  $f(x) = e^x$  at  $a = 0$ . Use this to estimate the value  $e^{0.01}$ .

13.2. Estimate the value of  $\sqrt[3]{26.8}$  using a linear approximation.

14. Given the position function of a particle, you are supposed to be able to compute its velocity, acceleration, understanding its physical meaning. You should be able to determine when a particle is speeding up or down, whether it is accelerating or decelerating. You are also supposed to be able to compute the total distance travelled during the given period.

**Example Problems**

14.1. The position of a particle is given by the function  $s = f(t) = t^3 - 6t^2 + 9t$ . Find the total distance traveled during the first 6 seconds.

14.2. A rock is thrown upward so that its height (in ft) after  $t$  seconds is given by  $h(t) = 48t - 16t^2$ . What is the velocity of the rock when its height is 32 ft on its way up ?

15. TWO "Related Rates" problems will be given in the Final Exam. Of particular importance are:

- Light house problem
- Inverted circular conical tank problem
- Kite problem
- Ladder problem
- Fred's swimming race problem
- Area (of a rectangle or a triangle) problem

15.1. A rectangle initially has dimensions 3 cm by 7 cm. All sides begin increasing in length at a rate of 2 cm/sec.

At what rate is the area of the rectangle increasing after 3 sec ?

15.2. The length of a rectangle is increasing at a rate of 5 cm/sec and its width is increasing at a rate of 4 cm/sec.

When the length is 30 cm and the width is 25 cm, how fast is the area of the rectangle increasing ?

15.3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of  $2 \text{ m}^3/\text{min}$ , find the rate at which the water level is rising when the water is 3 m deep.

15.4. (speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

15.5 (angle) A 15-foot plank of wood is leaning against a vertical wall and its bottom is being pushed toward the wall at the rate of 2 ft/sec.

At what rate is the angle  $\theta$  between the plank and the ground changing when the acute angle the plank makes with the ground is  $\pi/4$ ?

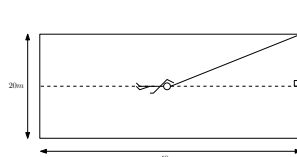
15.6. A lighthouse is located on a small island 4 km away from the nearest point  $P$  on a straight shoreline and its light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from  $P$  ?

15.7. A kite 100 ft above the ground moves horizontally at a speed of 3 ft/sec. At what rate is the angle (in radians) between the string and the horizontal decreasing when 200 ft of string have been let out ?

15.8. Fred is swimming a race in a pool 40 m long and 20 m wide. He is swimming his race in the center lane so that he is always 10 m from either side. His coach is standing at the corner of the pool by the finish line. Suppose Fred swims his race at a constant rate of 2 m/s.

(i) Let  $\theta$  be the angle between the side of the pool and the line between Fred and his coach. What is the rate at which  $\theta$  is changing (in radians per second) when Fred is halfway through the 40m race?

(ii) Consider the distance between Fred and the coach. What is the rate at which the distance between Fred and his coach is decreasing when he is halfway through the race.



16. You are supposed to be able to determine the exact values of the formulas involving the trigonometric and inverse trigonometric functions. You should pay special attention to the range of the inverse trigonometric function. You need to know the double angle formulas for sine, cosine, and tangent.

### Example Problems

16.1. Find the exact values of the following expression.

$$(i) \tan \left( \sin^{-1} \left( \frac{4}{5} \right) \right), (ii) \sin^{-1} \left( \sin \left( \frac{7\pi}{3} \right) \right), (iii) \sin \left( 2 \sin^{-1} \left( \frac{12}{13} \right) \right)$$

17. You are supposed to know how to find the absolute maximum and absolute minimum of a function  $f$  defined on the closed interval  $[a, b]$ , by comparing the values on the end points  $f(a)$ ,  $f(b)$  and the values on the critical value(s)  $f(c)$ 's. You should know what the definition of a critical value is.

### Example Problems

17.1. Find the absolute maximum/minimum and local maximum/minimum of the function defined by

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on the closed interval  $[-1, 4]$ .

17.2. Find the absolute maximum and absolute minimum values of the function  $f$  on the given interval.

(a)  $f(x) = 2x^3 - 3x^2 - 12x + 1$  on  $[-2, 3]$

(b)  $f(x) = xe^{x/2}$  on  $[-3, 1]$

(c)  $f(x) = x\sqrt{32 - x^2}$  on  $[0, 5]$

(d)  $f(t) = 2 \cos t + \sin 2t$  on  $[0, \pi/2]$

(e)  $f(x) = \ln(x^2 + x + 1)$  on  $[-1, 1]$

18. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

### Example Problems

18.1. The first derivative of a function  $f$  is given by

$$f'(x) = (x + 2)^2(x + 1)(x - 1)^3(x - 3)^2(x - 5).$$

Find the values of  $x$  for which the function  $f$  takes

(a) local maximum, and

(b) local minimum.



18.2. Consider the function  $f(x) = x^8(x - 4)^7$ .

(a) Find the critical numbers of the function  $f$ .

(b) What does the Second Derivative Test tell you about the behavior of  $f$  at these critical numbers ?

(c) What does the First Derivative Test tell you that the Second Derivative test does not ?

18.3. How many inflection points does the graph of the function  $y = f(x) = x^5 - 5x^4 + 25x$  have ?

18.4. We have a function whose first derivative is given by the formula  $f'(x) = (x - 1)^3(x + 1)^3$ . Find the  $x$ -coordinates of the local extrema and the inflection points of the function.

18.5. Which of the following statements are ALWAYS true for a function which is differentiable over  $(-\infty, \infty)$  ?

1. If  $f'(x) < 0$  for  $x < 0$  and  $f'(0) = 0$ , then  $f$  has a local minimum at  $x = 0$ .

2. If  $f$  has a local maximum at  $x = 0$ , then  $f'(0) = 0$ .

3. If  $f'(x) < 0$  for  $x < 0$  and  $f'(x) > 0$  for  $x > 0$ , then  $f$  has an absolute minimum at  $x = 0$ .

4. If  $f'(0) = 0$  and  $f''(0) < 0$ , then  $f$  has an absolute maximum at  $x = 0$ .

19. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form  $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$ .

### Example Problems

19.1. Compute the following limits:

(a)  $\lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}}$

(b)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}$

(c)  $\lim_{x \rightarrow 0} \frac{\sin x}{1 - x^2}$

(d)  $\lim_{x \rightarrow 0} \frac{3x - 2^x}{7^x - 6^x}$

(e)  $\lim_{x \rightarrow \pi} \frac{e^{\sin x} - 1}{x - \pi}$

20. You are suppose to know how to compute the limits of the form  $\pm\infty \times 0, \infty - \infty$ .

### Example Problems

20.1. Compute the following limits:

(a)  $\lim_{x \rightarrow 0^+} \sin(x) \cdot \ln(2x)$

(b)  $\lim_{x \rightarrow \infty} 2x \cdot \tan\left(\frac{1}{3x}\right)$

(c)  $\lim_{x \rightarrow 0^+} \cos x \cdot \arctan\left(\frac{3}{x}\right)$

(d)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x + 3} - x)$

(e)  $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

21. You are supposed to be able to compute the limits  $\lim_{x \rightarrow a} [f(x)]^{g(x)}$  formally of the form  $0^0, \infty^0, 1^\infty$ .

### Example Problems

21.1. Compute the following limits:

(a)  $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{7x}$

(b)  $\lim_{x \rightarrow \infty} \left(\frac{2x+1}{2x-1}\right)^{4x+1}$

(c)  $\lim_{x \rightarrow \infty} (2x + e^{5x})^{1/x}$

(d)  $\lim_{x \rightarrow 0^+} \tan(5x)^{\sin x}$

(e)  $\lim_{x \rightarrow 0} (1 + 3x)^{1/x}$

22. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down). You are also supposed to be able to determine the horizontal/vertical/slant asymptotes of the graph.

### Example Problems

22.1. Draw the graph of the following function:

(a)  $y = f(x) = \frac{x}{x^2 - 16}$

(b)  $y = f(x) = \frac{x}{x^2 + 16}$

(c)  $y = f(x) = \frac{1}{x^2 - 16}$

(d)  $y = f(x) = \frac{x^2}{x^2 - 16}$

(e)  $y = f(x) = e^{-x} \sin x$  on  $[0, 2\pi]$

(f)  $y = f(x) = \ln|x^2 - 10x + 24|$

22.2. Find the horizontal/vertical asymptote(s) of the following functions

$$(i) y = f(x) = \frac{x^3 + 4x^2 + x - 6}{x(x^2 - 1)}.$$

$$(ii) y = f(x) = \frac{x^2 - x}{x^2 - 4x + 3}.$$

$$(iii) y = f(x) = \frac{3e^x}{e^x - 1}.$$

$$(iv) y = f(x) = \frac{x - 5}{\sqrt{x^2 - 5x}}.$$

$$(v) y = f(x) = \frac{(x - 5)(x + 2)}{(x + 1)\sqrt{x^2 - 5x}}.$$

22.3. Find the equations of the horizontal, vertical, and slant asymptotes of the function

$$f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}.$$

23. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply this corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise: If  $f'(x) = 0$  for all values of  $x \in (a, b)$ , then a continuous function  $f$  on the closed interval  $[a, b]$  is actually a constant.

### Example Problems

23.1. Consider the function  $f(x) = x^3 - x$  over the interval  $[0, 2]$ . Does it satisfy the conditions for the Mean Value Theorem to hold? If it does, find the value(s)  $c \in (0, 2)$  such that  $f'(c) = \frac{f(2) - f(0)}{2 - 0}$ .

23.2. Consider the function  $y = f(x) = x^{2/3}$  over the interval  $[-1, 1]$ . Does it satisfy the conditions for the Mean Value Theorem to hold? Do we have any value  $c \in (-1, 1)$  such that  $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$ ?

23.3. Suppose that the function  $f$  is continuous on the interval  $[-2, 7]$  and differentiable on  $(-2, 7)$ . We have  $f(2) = 4$  and  $f(4) = 6$ . Applying the Mean Value Theorem, what can you conclude?

23.4. Determine the exact value of

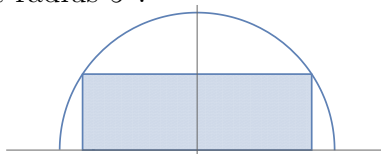
$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(7).$$

24. TWO Optimization Problems will be given in the Final Exam.

### Example Problems

- Maximize the area of a rectangle inscribed in a (semi-)circle and of a rectangle whose vertices are on the specified graph
- Maximize the area of a rectangle inside of a right triangle
- Minimize the cost of making a cylindrical can
- Hinge problem

24.1. What is the largest area of the rectangle inscribed in a semi-circle of radius 1 ? How about radius 5 ?



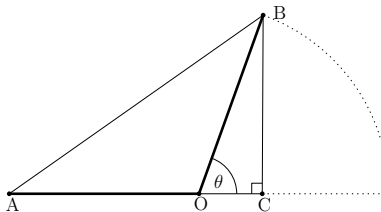
What is the largest area of the rectangle inscribed in the ellipse defined by the equation  $\frac{x^2}{7^2} + \frac{y^2}{3^2} = 1$  ?

24.2. What is the area of the largest rectangle that can fit in the region above the  $x$ -axis and below the curve  $y = \frac{1}{1+x^2}$  ?



24.3. A cylindrical can (with a top and bottom) is to be made to hold 1 L = 1,000 cm<sup>3</sup> of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can. What happens to the answer if the condition is changed so that the cylindrical can (with a top and bottom) is to be made to hold 3 L = 3,000 cm<sup>3</sup> of oil.

24.4. Two wooden bars of equal length  $AO = BO = 1$  ft are connected by a hinge at point  $O$  so that one can rotate the bar  $BO$  around as shown in the picture below. Find the maximum area of the triangle  $\triangle ABC$  when  $0 < \theta < \pi$ .



### New materials after Exam 3

25. You are supposed to know how to compute the integral using the Riemann Sum. Conversely, you should know how to compute the limit in the form of Riemann sum using the integration and Fundamental Theorem of Calculus. You are also supposed to know how to compute the integral knowing its geometrical meaning.

#### Example Problems

25.1. Write down the formula for approximating the integration  $\int_0^1 \sqrt{1-x^2} dx$  as the Riemann sum dividing the interval  $[0, 1]$  into  $n$  equal subintervals and using the left end points.

25.2. Compute the following limits:

$$(i) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \sqrt{3 + i \cdot \frac{5}{n}} \right) \cdot \frac{5}{n}$$

$$(ii) \lim_{n \rightarrow \infty} \sum_{i=1}^{3n} \left( 5 + i \cdot \frac{7}{3n} \right)^5 \cdot \frac{2}{3n}$$

$$(iii) \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \frac{n}{n^2 + i^2} \right)$$

HINT: Use the equation

$$\sum_{i=1}^n \frac{n}{n^2 + i^2} = \sum_{i=1}^n \frac{1}{1 + \left(\frac{i}{n}\right)^2} \cdot \frac{1}{n}$$

$$(iv) \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \frac{1}{\sqrt{4n^2 - i^2}} \right)$$

HINT: Use the equation

$$\sum_{i=1}^n \frac{1}{\sqrt{4n^2 - i^2}} = \sum_{i=1}^n \frac{1}{\sqrt{1 - \left(\frac{i}{2n}\right)^2}} \cdot \frac{1}{2n}$$

26. You are supposed to understand the meaning of the Fundamental Theorem of Calculus, and use it to compute the derivative of a function given in the form of an integration. You are also supposed to understand the special features of the integration involving even/odd functions.

**Example Problems**

26.1. Compute the following.

(i)  $\frac{d}{dx} \left( \int_0^x \sqrt{1+t^2} dt \right)$

(ii)  $\frac{d}{dx} \left( \int_0^{x^4} \sec(t) dt \right)$

(iii)  $\frac{d}{dx} \left( \int_{2x}^{3x} \frac{t^2-1}{t^2+1} dt \right)$

26.2. What value of  $b$  ( $> -1$ ) maximizes the integral

$$\int_{-1}^b x^2(5-x)dx ?$$

26.3. Compute the following integrals.

(i)  $\int_{\pi/4}^{\pi/3} \sec^2(x) dx$

(ii)  $\int_0^4 2^x dx$

(iii)  $\int_{-2}^2 (1 - |x|^3) dx$

(iv)  $\int_{-1}^1 \frac{\tan(x)}{1+x^2+x^4} dx$

26.4. If  $f(x) = \int_0^{\sin(x)} \sqrt{1+t^2} dt$  and  $g(y) = \int_3^y f(x) dx$ , find  $g''(\pi/6)$ .

26.5. Find all the points on  $[-\sqrt{5}, \sqrt{5}]$  at which  $f(x) = x^2 - 5$  equals its average value over  $[-\sqrt{5}, \sqrt{5}]$ .

26.6. Set  $F(x) = \int_{x^2}^{x^3} \sqrt{\ln t} dt$ . Find  $F'(e)$ .

27. You are supposed to understand that the differential equation  $\frac{dy}{dx} = ky$  has a solution of the form  $y = y(0)e^{kt}$ , and should be able to apply it to analyze the population growth and radioactive decay. In the case of the radioactive decay, you should also understand the formula  $m(t) = m(0)2^{-t/h}$  in terms of the half-life  $h$ .

**Example Problems**

27.1. A culture of a single cell creature Amoeba is found to triple its population in three weeks. Find its relative growth rate  $k$ .

27.2. The number of bacteria in a cell culture is initially observed to be 50. Three hours later the number is 100. Assuming that the bacteria

grow exponentially, how many hours after the initial observation does the number of bacterial become equal to 700?

27.3. The half-life of cesium-137 is 30 years. Suppose we have a 60-mg sample at the beginning. How long will it take until the remain of the sample becomes 1-mg ?

27.4. A parchment fragment was discovered that had about 74% as much  $^{14}\text{C}$  radioactivity as does plant material on Earth today. Estimate the age of the parchment, knowing that the half-life of  $^{14}\text{C}$  is 5730 years.

27.5. Initially there was 15 gm of a radioactive substance. After 3 years, only 7 gm remained. What is the half-life of this substance ?

28. You are supposed to know how to use the Substitution Rule to compute the indefinite and/or definite integrals.

### Example Problems

28.1. Compute the following integrals:

(i)  $\int \tan x \, dx$

(ii)  $\int \frac{\ln x}{x} \, dx$

(iii)  $\int_0^4 \sqrt{1+2x} \, dx$

(iv)  $\int_0^2 x^5 \sqrt{1+x^2} \, dx$

(v)  $\int_0^{\pi/4} \sec^4 x \tan x \, dx$

(vi)  $\int_0^{\pi/4} \tan^3 x \sec^2 x \, dx$

(vii)  $\int_{-\pi/4}^{\pi/4} \tan^3 x \sec^2 x \, dx$

(viii)  $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx$

(ix)  $\int_{\pi/6}^{\pi/3} \tan x \, dx$