MA 26100 - Fall 2019

Study Guide # 2

1. Constrained extreme values via **Lagrange Multipiers:** Max/min -ize $f(\mathbf{v})$ subject to constraint $g(\mathbf{v}) = C$, solve the system $\nabla f = \lambda \nabla g$ and $g(\mathbf{v}) = C$.

2. Double integrals; Double Riemann sums: $\displaystyle\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\ \Delta A$;

3. Type I region $R: \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $R: \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and $\displaystyle\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Polar: $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$, $\tan\theta = \dfrac{y}{x}$ (make sure θ in correct quadrant).

Change of Variables Formula in Polar Coordinates: if $R: \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\underset{\uparrow}{r}\,dr\,d\theta.$$

5. Applications of double integrals:

(a) Area of region R is $A(R) = \displaystyle\iint_R dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $V = \displaystyle\iint_R f(x,y)\,dA$

(c) Mass of R is $m = \displaystyle\iint_R \rho(x,y)\,dA$, where $\rho(x,y) =$ density (per unit area).

(d) Moment about the x-axis $M_x = \displaystyle\iint_R y\,\rho(x,y)\,dA$; moment about the y-axis $M_y = \displaystyle\iint_R x\,\rho(x,y)\,dA$.

(e) Center of mass $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\displaystyle\iint_R x\,\rho(x,y)\,dA}{\displaystyle\iint_R \rho(x,y)\,dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\displaystyle\iint_R y\,\rho(x,y)\,dA}{\displaystyle\iint_R \rho(x,y)\,dA}$

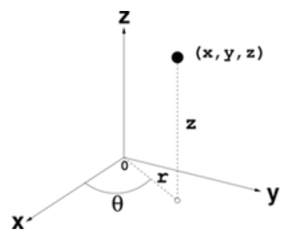
6. Elementary solids $D \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids D:

$$\iiint\limits_D f(x,y,z)\,dV = \iint\limits_R \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dA \text{ for } D = \{(x,y) \in R,\ u(x,y) \le z \le v(x,y)\};$$

volume of solid D is $V(D) = \iiint\limits_D dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

7. Cylindrical Coordinates (r, θ, z):

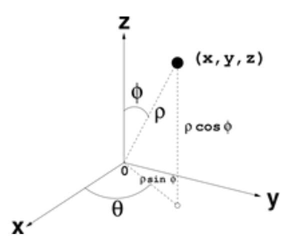
$$\text{From CC to RC :} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

8. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

$$\text{From SC to RC :} \quad \begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

9. Triple integrals in <u>Cylindrical Coordinates</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint\limits_D f(x,y,z)\ dV = \iiint\limits_D f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

$\uparrow$

10. Triple integrals in <u>Spherical Coordinates</u>: $\begin{cases} x = (\rho \sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi \;\; d\rho\, d\phi\, d\theta}$

$$\iiint_D f(x,y,z)\ dV = \iiint_D f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\underset{\uparrow}{\rho^2}\sin\phi\, d\rho\, d\phi\, d\theta$$

11. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\mathbf{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$ and
$\mathbf{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$;
$\mathbf{F}$ is a conservative vector field if $\mathbf{F} = \nabla f$, for some real-valued function f (potential).

12. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t)$, $y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),\ y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\displaystyle\int_C f(x,y)\,ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\displaystyle\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

(c) $\displaystyle\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$