Study Guide # 3

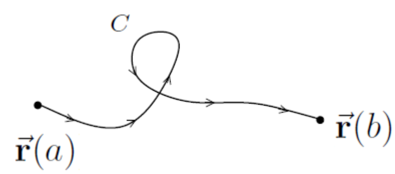
You also need Study Guides # 1 and # 2 for the Final Exam

0. Line integral of a vector field $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\,\vec{\mathbf{k}}$ along an *oriented* curve C, parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ and $a \le t \le b$, is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right) ds = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\ dt$$

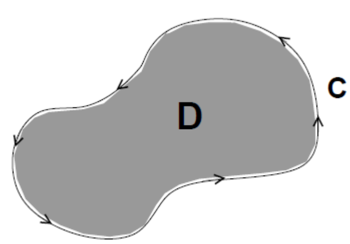
where $\vec{\mathbf{T}}(t) = \vec{\mathbf{r}}'/|\vec{\mathbf{r}}'|$ is the unit tangent vector.
(dependent of orientation of C, other properties and applications of line integrals of $\vec{\mathbf{F}}$)

1. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\displaystyle\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



2. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$;
how to determine a potential function f if $\vec{\mathbf{F}} = \nabla f$.

3. <u>Green's Theorem</u>: $\displaystyle\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ ($C =$ boundary of D):



4. Del Operator: $\nabla = \dfrac{\partial}{\partial x}\,\vec{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\vec{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

$$\text{curl}\ \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div}\ \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

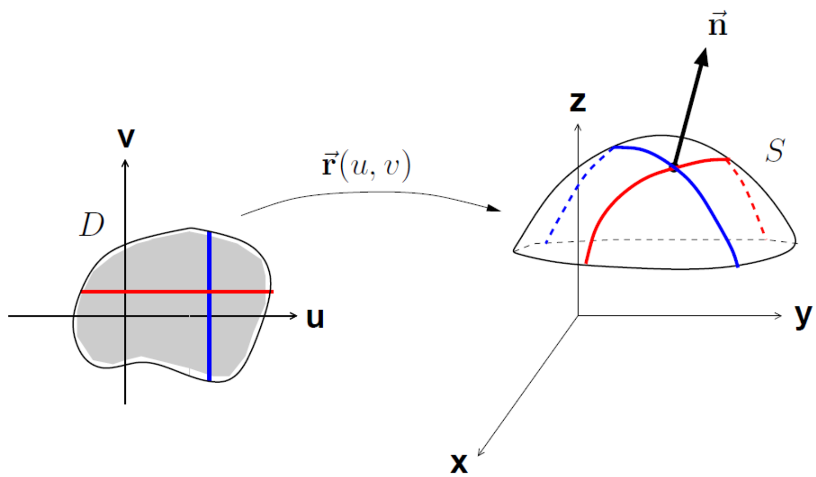
(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field ($\vec{\mathbf{F}} = \nabla f$) in a simply-connected domain.

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.

(iii) *Laplace's Equation*: $\nabla^2 f = \operatorname{div}(\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0.$

(iv) For functions with continuous partials, $\operatorname{curl}(\nabla f) = \vec{\mathbf{0}}$, and $\operatorname{div}(\operatorname{curl} \vec{\mathbf{F}}) = 0$.

5. Parametric surface S: $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where $(u, v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

6. Surface area of a surface S:

(i) $A(S) = \displaystyle\iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$

(ii) If S is the graph of $z = h(x, y)$ above D, then $A(S) = \displaystyle\iint_D \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2}\, dA$;

<u>Remark</u>: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$ = differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$

7. The surface integral of f over the surface S:

(i) $\displaystyle\iint_S f(x, y, z)\, dS = \iint_D f(\vec{\mathbf{r}}(u, v))\, |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA.$

(ii) If S is the graph of $z = h(x, y)$ above D, then

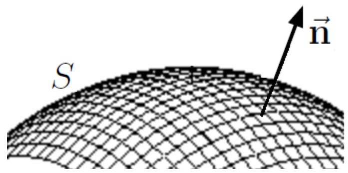
$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, h(x, y))\, \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2}\, dA.$$

8. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$):

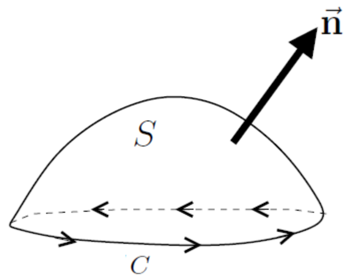
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS = \underline{\textbf{flux}} \text{ of } \vec{\mathbf{F}} \text{ across the surface } S.$$

If S is the graph of $z = h(x, y)$ above D, with $\vec{\mathbf{n}}$ oriented upward, and $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, then

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \left(-P\frac{\partial h}{\partial x} - Q\frac{\partial h}{\partial y} + R\right) dA$$



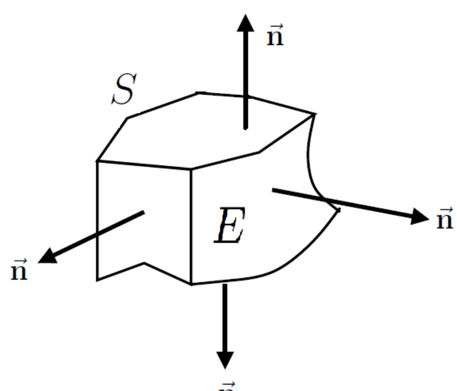
9. <u>Stokes' Theorem</u>: $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ (recall, $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}$).



$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} =$ <u>*circulation*</u> of $\vec{\mathbf{F}}$ around C.

10. <u>The Divergence Theorem/Gauss' Theorem</u>: $\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}} \; dV$

(recall, $\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}$).



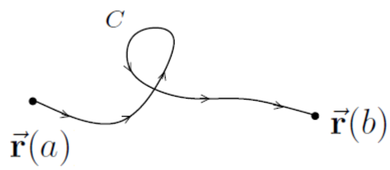
11. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t)$, where $a \le t \le b$	$S : \vec{\mathbf{r}}(u,v)$, where $(u,v) \in D$
$ds = \lvert \vec{\mathbf{r}}\,'(t) \rvert \, dt =$ differential of arc length	$dS = \lvert \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \rvert \, dA =$ differential of surface area
$\displaystyle\int_C ds =$ length of C	$\displaystyle\iint_S dS =$ surface area of S
$\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\,\lvert \vec{\mathbf{r}}\,'(t)\rvert\,dt$ (independent of orientation of C)	$\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\,\lvert \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\,dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t)\,dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$
$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt$ (depends on orientation of C)	$\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right)\ ds$ The *circulation* of $\vec{\mathbf{F}}$ around C	$\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right)\ dS$ The *flux* of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

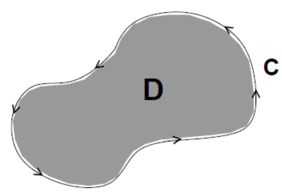
12. Integration Theorems:

<u>Fundamental Theorem of Calculus</u>: $\int_a^b F'(x)\,dx = F(b) - F(a)$

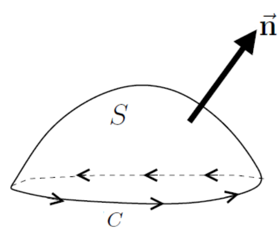
<u>Fundamental Theorem of Calculus For Line Integrals</u>: $\int_a^b \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$



<u>Green's Theorem</u>: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$



<u>Stokes' Theorem</u>: $\iint_S \operatorname{curl}\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$



<u>Divergence Theorem</u>: $\iiint_E \operatorname{div}\vec{\mathbf{F}}\;dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$