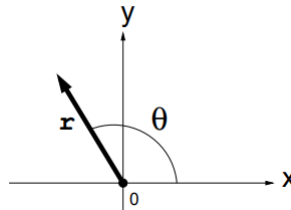


Study Guide # 1

1. Vectors in \mathbb{R}^2 and \mathbb{R}^3

(a) $\vec{v} = \langle a, b, c \rangle = a\vec{i} + b\vec{j} + c\vec{k}$; vector addition and subtraction geometrically using parallelograms spanned by \vec{u} and \vec{v} ; length or magnitude of $\vec{v} = \langle a, b, c \rangle$, $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$; directed vector from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$ given by $\vec{v} = \overline{P_0P_1} = P_1 - P_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

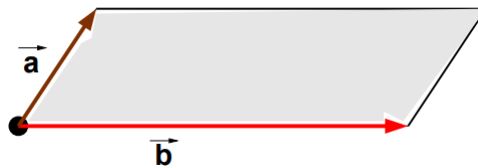
(b) Dot (or inner) product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$; properties of dot product; useful identity: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$; angle between two vectors \vec{a} and \vec{b} : $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$; $\vec{a} \perp \vec{b}$ if and only if $\vec{a} \cdot \vec{b} = 0$; the vector in \mathbb{R}^2 with length r with angle θ is $\vec{v} = \langle r \cos \theta, r \sin \theta \rangle$:



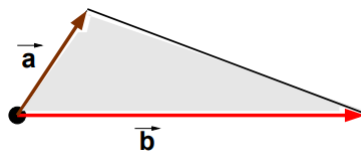
(c) Cross product (only for vectors in \mathbb{R}^3):

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

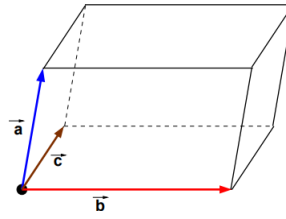
properties of cross products; $\vec{a} \times \vec{b}$ is perpendicular (orthogonal or normal) to both \vec{a} and \vec{b} ; area of parallelogram spanned by \vec{a} and \vec{b} is $A = |\vec{a} \times \vec{b}|$:



the area of the triangle spanned is $A = \frac{1}{2} |\vec{a} \times \vec{b}|$:



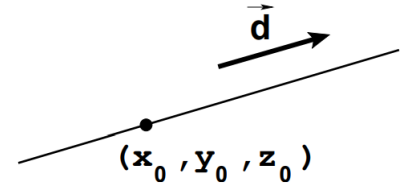
Volume of the parallelepiped spanned by \vec{a} , \vec{b} , \vec{c} is $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$:



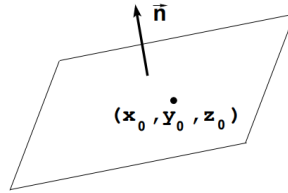
2. Equation of a line L through $P_0(x_0, y_0, z_0)$ with direction vector $\vec{v} = \langle a, b, c \rangle$:

Vector Form: $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \vec{v}$.

Parametric Form: $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$



3. Equation of the plane through the point $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ (\vec{n} is a *normal vector* to the plane) is $\langle (x - x_0), (y - y_0), (z - z_0) \rangle \cdot \vec{n} = 0$; Sketching planes (consider x, y, z intercepts).



4. Quadric surfaces (can sketch them by considering various *traces*, i.e., curves resulting from the intersection of the surface with planes $x = k$, $y = k$ and/or $z = k$); some generic equations have the form:

(a) *Ellipsoid*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(b) *Hyperboloid of One Sheet*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(c) *Hyperboloid of Two Sheets*: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(d) *Elliptic Paraboloid*: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

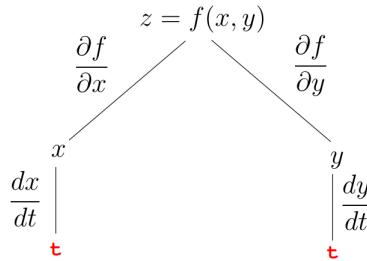
(e) *Hyperbolic Paraboloid (Saddle)*: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

(f) *Cone*: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

- 5.** Vector-valued functions $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$; tangent vector $\vec{r}'(t)$ for smooth curves, unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$; differentiation rules for vector functions, including:
- (i) $\{\phi(t) \vec{v}(t)\}' = \phi(t) \vec{v}'(t) + \phi'(t) \vec{v}(t)$, where $\phi(t)$ is a real-valued function
 - (ii) $(\vec{u} \cdot \vec{v})' = \vec{u} \cdot \vec{v}' + \vec{u}' \cdot \vec{v}$
 - (iii) $(\vec{u} \times \vec{v})' = \vec{u} \times \vec{v}' + \vec{u}' \times \vec{v}$
 - (iv) $\{\vec{v}(\phi(t))\}' = \phi'(t) \vec{v}'(\phi(t))$, where $\phi(t)$ is a real-valued function
- 6.** Integrals of vector functions $\int \vec{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle$; arc length of curve parameterized by $\vec{r}(t)$ is $L = \int_a^b |\vec{r}'(t)| dt$; arc length function $s(t) = \int_a^t |\vec{r}'(u)| du$; reparameterize by arc length: $\vec{\sigma}(s) = \vec{r}(t(s))$, where $t(s)$ is the inverse of the arc length function $s(t)$; the *curvature* of a curve parameterized by $\vec{r}(t)$ is $\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$. **Note:** $\sqrt{\alpha^2} = |\alpha|$.
- 7.** $\vec{r}(t)$ = position of a particle, $\vec{r}'(t) = \vec{v}(t)$ = velocity; $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ = acceleration; $|\vec{r}'(t)| = |\vec{v}(t)|$ = speed; Newton's 2nd Law: $\vec{F} = m \vec{a}$; motion in space $\vec{F} = \langle 0, 0, -mg \rangle$.
- 8.** Domain and range of a function $f(x, y)$ and $f(x, y, z)$; *level curves* (or contour curves) of $f(x, y)$ are the curves $f(x, y) = k$; using level curves to sketch surfaces; *level surfaces* of $f(x, y, z)$ are the surfaces $f(x, y, z) = k$.
- 9.** Limits of functions $f(x, y)$ and $f(x, y, z)$; limit of $f(x, y)$ does not exist if different paths approach to (a, b) yield different limits; continuity.
- 10.** Partial derivatives $\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$,
 $\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$; higher order partial derivatives: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$,
 $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$, etc; mixed partial derivatives.
- 11.** Equation of the tangent plane to the graph of $z = f(x, y)$ at (x_0, y_0, z_0) is given by
 $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- 12.** Total differential for $z = f(x, y)$ is $dz = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$; total differential for $w = f(x, y, z)$ is $dw = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$; linear approximation for $z = f(x, y)$ is given by $\Delta z \approx dz$, i.e., $f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, where $\Delta x = dx$, $\Delta y = dy$;
Linearization of $f(x, y)$ at (a, b) is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$;
 $L(x, y) \approx f(x, y)$ near (a, b) .

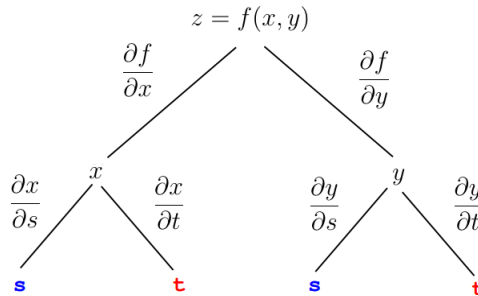
13. CHAIN RULE; different forms of the Chain Rule: Form 1, Form 2; CHAIN RULE (GENERAL FORM): Tree diagrams. For example:

(a) If $z = f(x, y)$ and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, then $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$:



(b) If $z = f(x, y)$ and $\begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases}$, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} :$$



etc.....

14. Implicit Differentiation:

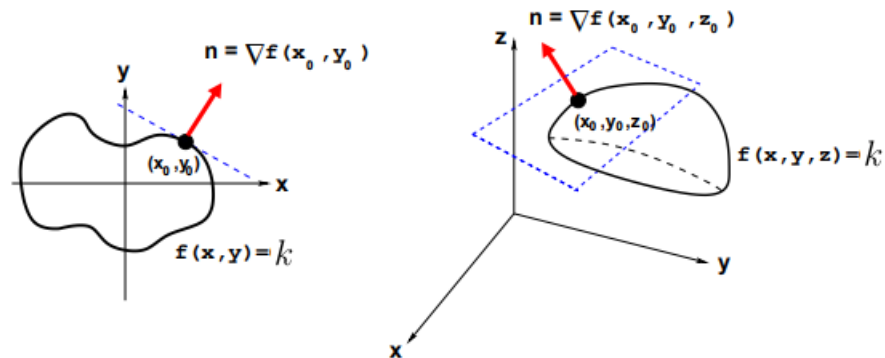
Part I: If $F(x, y) = 0$ defines y as function of x (i.e., $y = y(x)$), then to compute $\frac{dy}{dx}$, differentiate both sides of the equation $F(x, y) = 0$ w.r.t. x and solve for $\frac{dy}{dx}$.

If $F(x, y, z) = 0$ defines z as function of x and y (i.e. $z = z(x, y)$), then to compute $\frac{\partial z}{\partial x}$, differentiate the equation $F(x, y, z) = 0$ w.r.t. x (hold y fixed) and solve for $\frac{\partial z}{\partial x}$. For $\frac{\partial z}{\partial y}$, differentiate the equation $F(x, y, z) = 0$ w.r.t. y (hold x fixed) and solve for $\frac{\partial z}{\partial y}$.

Part II: If $F(x, y) = 0$ defines y as function of $x \implies \frac{dy}{dx} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y}$;

while if $F(x, y, z) = 0$ defines z as function of x and $y \implies \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial z}$ and $\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} / \frac{\partial F}{\partial z}$.

- 15.** Gradient vector for $f(x, y)$: $\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$, properties of gradients; gradient points in direction of maximum rate of increase of f , maximum rate of increase is $|\nabla f|$; $\nabla f(x_0, y_0) \perp$ level curve $f(x, y) = k$ and, in the case of 3 variables, $\nabla f(x_0, y_0, z_0) \perp$ level surface $f(x, y, z) = k$:



- 16.** Directional derivative of $f(x, y)$ at (x_0, y_0) in the direction \vec{u} : $D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$, where \vec{u} must be a *unit* vector; tangent planes to level surfaces $f(x, y, z) = k$ (a normal vector at (x_0, y_0, z_0) is $\vec{n} = \nabla f(x_0, y_0, z_0)$).

- 17.** Relative/local extrema; critical points (points where $\nabla f = \vec{0}$ or ∇f does not exist).

- 18.** 2nd Derivatives Test: Suppose the 2nd partials of $f(x, y)$ are continuous in a disk with center (a, b) and $\nabla f(a, b) = \vec{0}$. Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)}$.

- (a) If $D > 0$ and $f_{xx}(a, b) > 0 \implies f(a, b)$ is a local minimum value.
 (b) If $D > 0$ and $f_{xx}(a, b) < 0 \implies f(a, b)$ is a local maximum value.
 (c) If $D < 0 \implies f(a, b)$ is a *not* a local min or local max value. So (a, b) is a **saddle point** of f .

If $D = 0$ (or if $\nabla f(a, b)$ does not exist or f has more than 2 variables) the test gives no information and you need to do something else to determine if a relative extremum exists.

- 19.** Absolute extrema; Max-Min Problems.