MA 16600
Study Guide For Exam 2- Lessons 11 to 20, Excluding 15

(1) **Techniques of Integration** This was on Exam I, but it is important to review it to be able to do the integrals that appear after trigonometric substitutions.

(a) **Trig Integrals**: Integrals of the type \( \int \sin^m x \cos^n x \, dx \) and \( \int \tan^m x \sec^n x \, dx \)

Some useful trig identities:

(i) \( \sin^2 \theta + \cos^2 \theta = 1 \) and \( \tan^2 \theta + 1 = \sec^2 \theta \)

(ii) \( \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \) and \( \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \)

(iii) \( \sin 2\theta = 2 \sin \theta \cos \theta \)

Some useful trig integrals:

(i) \( \int \tan u \, du = \ln|\sec u| + C \)

(ii) \( \int \sec u \, du = \ln|\sec u + \tan u| + C \)

(b) **Trig integrals of the form**: \( \int \sin mx \sin nx \, dx \), \( \int \cos mx \cos nx \, dx \), \( \int \sin mx \cos nx \, dx \), use these trig identities:

\[
\sin A \sin B = \frac{1}{2} \{\cos(A - B) - \cos(A + B)\}
\]

\[
\cos A \cos B = \frac{1}{2} \{\cos(A - B) + \cos(A + B)\}
\]

\[
\sin A \cos B = \frac{1}{2} \{\sin(A - B) + \sin(A + B)\}
\]

(c) **Trigonometric Substitutions**:

<table>
<thead>
<tr>
<th>Expression*</th>
<th>Trig Substitution</th>
<th>Identity needed</th>
<th>Square root</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta )</td>
<td>( a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta )</td>
<td>( \sqrt{a^2 - x^2} = a \cos \theta )</td>
</tr>
<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan \theta )</td>
<td>( a^2 + a^2 \tan^2 \theta = a^2 \sec^2 \theta )</td>
<td>( \sqrt{a^2 + x^2} = a \sec \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta )</td>
<td>( a^2 \sec^2 \theta - a^2 = a^2 \tan^2 \theta )</td>
<td>( \sqrt{x^2 - a^2} = a \tan \theta )</td>
</tr>
</tbody>
</table>

* Or powers of these expressions.
(2) Integration via Partial Fractions: Use for (proper) rational functions $\frac{R(x)}{Q(x)}$;
If $\deg R(x) \geq \deg Q(x)$, i.e. rational function is improper, then do polynomial division before using partial fractions: $P(x) = S(x)Q(x) + R(x)$ where $\deg R(x) < \deg Q(x)$.

(3) Improper integrals: **Type I** (unbounded intervals) $\int_a^b f(x) \, dx$, $\int_{-\infty}^b f(x) \, dx$ or $\int_{-\infty}^{\infty} f(x) \, dx$;
Improper integrals of **Type II** (discontinuous integrand at one or both endpoints) $\int_a^b f(x) \, dx$.

$$\int_0^1 \frac{1}{x^p} \, dx \text{ converges if and only if } p < 1$$
$$\int_1^{\infty} \frac{1}{x^p} \, dx \text{ converges if and only if } p > 1.$$  

**Comparison Theorem:** Let $f(x)$ and $g(x)$ be continuous for $x \geq a$.
(a) If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^b g(x) \, dx$ converges $\implies \int_a^b f(x) \, dx$ also converges.
(b) If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and $\int_a^b g(x) \, dx$ diverges $\implies \int_a^b f(x) \, dx$ also diverges.

(4) Arc length $L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$ or $L = \int_c^d \sqrt{1 + (g'(y))^2} \, dy$.

(5) Surface area of revolution: $S = \int 2\pi \{\text{ribbon radius}\} \, ds$ or $S = \int 2\pi r \, ds$,
where $ds = \sqrt{1 + (f'(x))^2} \, dx$ or $ds = \sqrt{1 + (g'(y))^2} \, dy.$
(6) Center of mass of a system of discrete masses \( m_1, m_2, \ldots, m_n \) located at \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) is \((\bar{x}, \bar{y})\), where

\[
\bar{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k},
\]

\(M_x\) = moment of system about the \(x\)-axis; \(M_y\) = moment of system about the \(y\)-axis; \(M\) = total mass of the system.

(7) Moments, center of mass (center of mass = centroid if density \(\rho\) = constant).

(a) Lamina defined by \(y = f(x)\), \(a \leq x \leq b\) and \(\rho\) = constant:

\[
\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \rho f(x) \, dx}{\int_a^b \rho f(x) \, dx} = \frac{\int_a^b x f(x) \, dx}{\int_a^b f(x) \, dx},
\]

\[
\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2} \rho \{f(x)\}^2 \, dx}{\int_a^b \rho f(x) \, dx} = \frac{\int_a^b \frac{1}{2} \{f(x)\}^2 \, dx}{\int_a^b f(x) \, dx}.
\]

(b) Lamina between two curves by \(y = f(x), y = g(x)\), \(a \leq x \leq b\) and \(\rho\) = constant:

\[
\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \rho (f(x) - g(x)) \, dx}{\int_a^b \rho (f(x) - g(x)) \, dx} = \frac{\int_a^b x (f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx},
\]

\[
\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2} \rho \left(\{f(x)\}^2 - \{g(x)\}^2\right) \, dx}{\int_a^b \rho (f(x) - g(x)) \, dx} = \frac{\int_a^b \frac{1}{2} \left(\{f(x)\}^2 - \{g(x)\}^2\right) \, dx}{\int_a^b (f(x) - g(x)) \, dx}.
\]
(8) Sequences; limits of sequences; Limit Laws for Sequences; monotone sequences (increasing and decreasing); bounded sequences; **Monotone Sequence Theorem**.

(9) Additional useful limit theorems:

(a) **Theorem**: If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \), then \( \lim_{n \to \infty} a_n = L \).

(b) **Squeeze Theorem for Sequences**: If \( a_n \leq b_n \leq c_n \) for all \( n \geq N_0 \) with \( a_n \to L \) and \( c_n \to L \), then \( b_n \to L \).

(c) **Theorem**: If \( a_n \to L \) and \( f \) is continuous at \( L \), then \( f(a_n) \to f(L) \).

(10) Infinite series \( \sum_{n=1}^{\infty} a_n \); \( n^{th} \) partial sum \( s_n = \sum_{k=1}^{n} a_k \); the infinite series \( \sum_{n=1}^{\infty} a_n \) **converges** to \( s \) if \( s_n \to s \); the infinite series **diverges** if \( \{s_n\} \) does not have a limit.

(11) **Divergence Test for Series**: If \( \lim_{n \to \infty} a_n \neq 0 \) or limit fails to exist \( \implies \sum_{n=1}^{\infty} a_n \) DIVERGES.

(12) Special Infinite Series:

(a) **Geometric Series**: \( \sum_{n=1}^{\infty} ar^{n-1} \)

(i) \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = a\left(1 + r + r^2 + r^3 + \cdots\right) = \frac{a}{1 - r} \), if \( |r| < 1 \).

(ii) \( \sum_{n=1}^{\infty} ar^{n-1} \) will DIVERGE if \( |r| \geq 1 \).

(13) Besides reviewing all homework problems, students should do the following review exercises from the book:

Section 7.3: 4, 5, 6, 8, 9, 10, 11, 12, 14, 19, 23
Section 7.4: 1, 2, 3, 4b, 9, 10, 11, 16, 19
Section 7.8: 3, 4, 8, 9, 13, 19, 21, 22, 26, 37
Section 8.1: 1, 11, 12, 13, 17
Section 8.2: 7, 8, 12, 13, 14, 18
Section 8.3: 29, 20, 31, 32, 33, 35
Section 11.1: 19, 20, 21, 22, 23, 24, 30, 31
Section 11.2: 21, 22, 23, 24, 25, 32

**Acknowledgement**: These notes written by Prof. Johnny Brown.