Spring 2019

MA 16600

Study Guide For Exam 2- Lessons 11 to 20, Excluding 15

(1) TECHNIQUES OF INTEGRATION This was on Exam I, but it is important to review it to be able to do the integrals that appear after trigonometric substitutions.

(a) **Trig Integrals**: Integrals of the type $\int \sin^m x \cos^n x\, dx$ and $\int \tan^m x \sec^n x\, dx$

Some useful trig identities:

(i) $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$

(ii) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ and $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(iii) $\sin 2\theta = 2\sin\theta\,\cos\theta$

Some useful trig integrals:

(i) $\int \tan u\ du = \ln|\sec u| + C$

(ii) $\int \sec u\ du = \ln|\sec u + \tan u| + C$

(b) **Trig integrals of the form**: $\int \sin mx\ \sin nx\ dx$, $\int \cos mx\ \cos nx\ dx$, $\int \sin mx\ \cos nx\ dx$, use these trig identities:

$$\sin A\ \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

$$\cos A\ \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

$$\sin A\ \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

(c) **Trigonometric Substitutions**:

*Expression**	*Trig Substitution*	*Identity needed*	*Square root*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$	$\sqrt{a^2-x^2} = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$	$\sqrt{a^2+x^2} = a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$	$\sqrt{x^2-a^2} = a\tan\theta$

* Or powers of these expressions.

(2) Integration via Partial Fractions: Use for (proper) rational functions $\dfrac{R(x)}{Q(x)}$;
If $\deg R(x) \geq \deg Q(x)$, i.e. rational function is improper, then do polynomial division before using partial fractions: $P(x) = S(x)\,Q(x) + R(x)$ where $\deg R(x) < \deg Q(x)$.

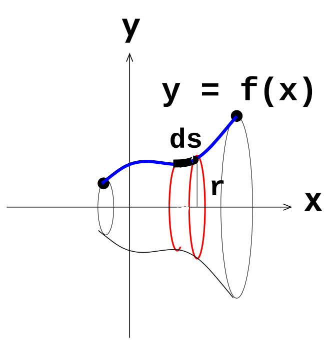
(3) Improper integrals: **Type I** (unbounded intervals) $\int_a^\infty f(x)\,dx\,, \int_{-\infty}^b f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$;
Improper integrals of **Type II** (discontinuous integrand at one or both endpoints) $\int_a^b f(x)\,dx$.

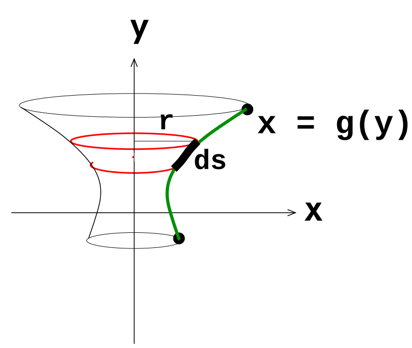
$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges if and only if } p < 1$$
$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if and only if } p > 1.$$

Comparison Theorem: Let $f(x)$ and $g(x)$ be continuous for $x \geq a$.
(a) If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \int_a^\infty f(x)\,dx$ also converges.
(b) If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and $\int_a^\infty g(x)\,dx$ diverges $\Longrightarrow \int_a^\infty f(x)\,dx$ also diverges.

(4) Arc length $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$ or $L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy$.

(5) Surface area of revolution: $S = \int 2\pi\,\{\texttt{ribbon radius}\}\,ds$ or $S = \int 2\pi r\,ds$,
where $ds = \sqrt{1 + (f'(x))^2}\,dx$ or $ds = \sqrt{1 + (g'(y))^2}\,dy$.





(6) Center of mass of a system of discrete masses $m_1, m_2, \cdots, m_n$ located at $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}, \qquad \overline{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k}$$

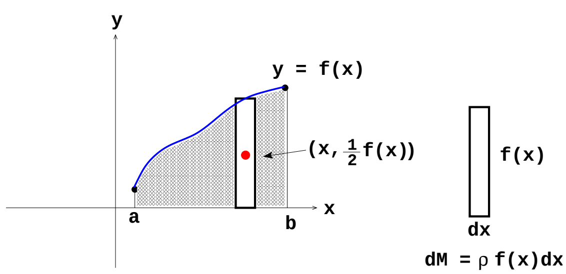
$M_x =$ moment of system about the $x-$axis; $M_y =$ moment of system about the $y-$axis; $M =$ total mass of the system.

(7) Moments, center of mass (center of mass = *centroid* if density $\rho =$ constant).

(a) Lamina defined by $y = f(x)$, $a \le x \le b$ and $\rho =$ constant:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x\rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

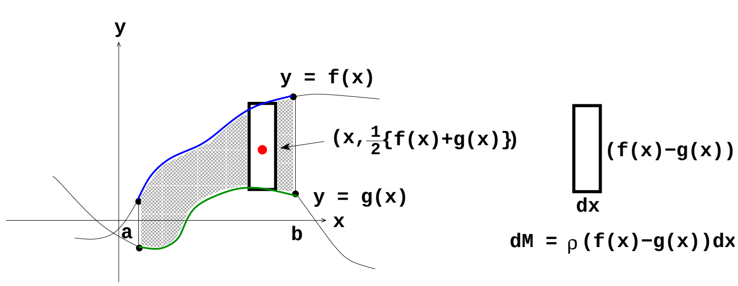
$$\overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\rho \{f(x)\}^2\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b \frac{1}{2} \{f(x)\}^2\,dx}{\int_a^b f(x)\,dx}$$



(b) Lamina between two curves by $y = f(x)$, $y = g(x)$, $a \le x \le b$ and $\rho =$ constant:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x\rho(f(x) - g(x))\,dx}{\int_a^b \rho(f(x) - g(x))\,dx} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))\,dx}$$

$$\overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\rho \left(\{f(x)\}^2 - \{g(x)\}^2\right)\,dx}{\int_a^b \rho(f(x) - g(x))\,dx} = \frac{\int_a^b \frac{1}{2} \left(\{f(x)\}^2 - \{g(x)\}^2\right)\,dx}{\int_a^b (f(x) - g(x))\,dx}$$



(8) Sequences; limits of sequences; Limit Laws for Sequences; monotone sequences (increasing and decreasing); bounded sequences; `Monotone Sequence Theorem`.

(9) Additional useful limit theorems:

(a) *Theorem*: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.

(b) *Squeeze Theorem for Sequences*: If $a_n \leq b_n \leq c_n$ for all $n \geq N_0$ with $a_n \longrightarrow L$ and $c_n \longrightarrow L$, then $b_n \longrightarrow L$.

(c) *Theorem*: If $a_n \longrightarrow L$ and f is continuous at L, then $f(a_n) \longrightarrow f(L)$.

(10) Infinite series $\sum_{n=1}^{\infty} a_n$; n^{th} partial sum $s_n = \sum_{k=1}^{n} a_k$; the infinite series $\sum_{n=1}^{\infty} a_n$ **converges** to s if $s_n \to s$; the infinite series **diverges** if $\{s_n\}$ does not have a limit.

(11) `Divergence Test for Series`: If $\lim_{n\to\infty} a_n \neq 0$ or limit fails to exist $\Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(12) Special Infinite Series:

(a) `Geometric Series`: $\sum_{n=1}^{\infty} ar^{n-1}$

(i) $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = a(1 + r + r^2 + r^3 + \cdots) = \frac{a}{1-r}$, **if** $|r| < 1$.

(ii) $\sum_{n=1}^{\infty} ar^{n-1}$ will DIVERGE **if** $|r| \geq 1$.

(13) Besides reviewing all homework problems, students should do the following review exercises from the book:
Section 7.3: 4, 5, 6, 8, 9, 10, 11, 12, 14, 19, 23
Section 7.4: 1, 2, 3, 4b, 9, 10, 11, 16, 19
Section 7.8: 3, 4, 8, 9, 13, 19, 21, 22, 26, 37
Section 8.1: 1, 11, 12, 13, 17
Section 8.2: 7, 8, 12, 13, 14, 18
Section 8.3: 29, 20, 31, 32, 33, 35
Section 11.1: 19, 20, 21, 22, 23, 24, 30, 31
Section 11.2: 21, 22, 23, 24, 25, 32

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