MA 26100 - Spring 2019

Study Guide # 2

1. Constrained extreme values via **Lagrange Multipiers:** Max/min -ize $f(\mathbf{v})$ subject to constraint $g(\mathbf{v}) = C$, solve the system $\nabla f = \lambda \nabla g$ and $g(\mathbf{v}) = C$.

2. Double integrals; Double Riemann sums: $\displaystyle\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\ \Delta A$;

3. Type I region $D : \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $D : \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\displaystyle\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and

$\displaystyle\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities: $\displaystyle mA \le \iint_D f(x,y)\,dA \le MA$, where $A =$ area of D and $m \le f(x,y) \le M$ on D.

5. Polar: $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$, $\tan\theta = \dfrac{y}{x}$ (make sure θ in correct quadrant).

Change of Variables Formula in Polar Coordinates: if $D : \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\underset{\uparrow}{r}\,dr\,d\theta.$$

6. Applications of double integrals:

(a) Area of region D is $\displaystyle A(D) = \iint_D dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $\displaystyle V = \iint_D f(x,y)\,dA$

(c) Mass of D is $\displaystyle m = \iint_D \rho(x,y)\,dA$, where $\rho(x,y) =$ density (per unit area); sometimes write $\displaystyle m = \iint_D dm$, where $dm = \rho(x,y)\,dA$.

(d) Moment about the x-axis $\displaystyle M_x = \iint_D y\,\rho(x,y)\,dA$; moment about the y-axis $\displaystyle M_y = \iint_D x\,\rho(x,y)\,dA$.

(e) Center of mass $(\overline{x}, \overline{y})$, where $\displaystyle \overline{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, $\displaystyle \overline{y} = \frac{M_x}{m} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$

(f) Surface Area $\displaystyle A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA$

7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E:
$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\;dA$ for $E = \{(x,y) \in D,\; u(x,y) \le z \le v(x,y)\}$;
volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

8. **Cylindrical Coordinates** (r, θ, z):

$$\text{From CC to RC :} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

9. **Spherical Coordinates** (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

$$\text{From SC to RC :} \quad \begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$$

Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

10. Triple integrals in <u>Cylindrical Coordinates</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint_E f(x,y,z)\;dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

11. Triple integrals in <u>Spherical Coordinates</u>: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2\sin\phi\;\;d\rho\,d\phi\,d\theta}$

$$\iiint_E f(x,y,z)\;dV = \iiint_E f(\rho\sin\phi\cos\theta,\; \rho\sin\phi\sin\theta,\; \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\mathbf{F}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$ and
$\mathbf{F}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle$;
$\mathbf{F}$ is a conservative vector field if $\mathbf{F} = \nabla f$, for some real-valued function f (potential).

13. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x,y)\, ds = \int_a^b f(x(t),\, y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

Remarks:

(a) $\displaystyle\int_C f(x,y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\displaystyle\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

(c) $\displaystyle\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

14. Line integral of vector field $\mathbf{F}(x,y)$ along C, parameterized by $\mathbf{r}(t)$ and $a \leq t \leq b$, is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of $\mathbf{F}$)

15. Connection between line integral of vector fields and line integral of functions:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T})\, ds$$

where $\mathbf{T}$ is the unit tangent vector to the curve C.

16. If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x,y)\, dx + Q(x,y)\, dy$; Work $= \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.