Study Guide # 2

1. Constrained extreme values via **Lagrange Multipliers**: Maximize \( f(v) \) subject to constraint \( g(v) = C \), solve the system \( \nabla f = \lambda \nabla g \) and \( g(v) = C \).

2. Double integrals; Double Riemann sums: \( \iiint_R f(x,y) \, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \, \Delta A \);

3. Type I region \( D : \{ g_1(x) \leq y \leq g_2(x) \} \); Type II region \( D : \{ h_1(y) \leq x \leq h_2(y) \} \);

iterated integrals over Type I and II regions:

\[
\iint_D f(x,y) \, dA = \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx \quad \text{and} \quad \iint_D f(x,y) \, dA = \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy,
\]

respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities:

\[ mA \leq \iint_D f(x,y) \, dA \leq MA, \]

where \( A = \text{area of } D \) and \( m \leq f(x,y) \leq M \) on \( D \).

5. Polar: \( r^2 = x^2 + y^2, x = r \cos \theta, y = r \sin \theta, \tan \theta = \frac{y}{x} \) (make sure \( \theta \) in correct quadrant).

Change of Variables Formula in Polar Coordinates: if \( D : \{ h_1(\theta) \leq r \leq h_2(\theta) \} \), then

\[
\iint_D f(x,y) \, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

6. Applications of double integrals:

(a) Area of region \( D \) is \( A(D) = \iint_D \, dA \)

(b) Volume of solid under graph of \( z = f(x,y) \), where \( f(x,y) \geq 0 \), is \( V = \iint_D f(x,y) \, dA \)

(c) Mass of \( D \) is \( m = \iint_D \rho(x,y) \, dA \), where \( \rho(x,y) \) = density (per unit area); sometimes write \( m = \iint_D dm \), where \( dm = \rho(x,y) \, dA \).

(d) Moment about the \( x \)-axis \( M_x = \iint_D y \, \rho(x,y) \, dA \); moment about the \( y \)-axis \( M_y = \iint_D x \, \rho(x,y) \, dA \).

(e) Center of mass \( (\bar{x}, \bar{y}) \), where \( \bar{x} = \frac{M_y}{m} = \frac{\iint_D x \, \rho(x,y) \, dA}{\iint_D \rho(x,y) \, dA}, \quad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y \, \rho(x,y) \, dA}{\iint_D \rho(x,y) \, dA} \)

(f) Surface Area \( A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA \)
7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids $E$:
$$\iiint_E f(x, y, z) \, dV = \iint_D \int_{u(x, y)}^{v(x, y)} f(x, y, z) \, dz \, dA \quad \text{for } E = \{(x, y) \in D, \ u(x, y) \leq z \leq v(x, y)\};$$
volume of solid $E$ is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes $M_{xy}$, $M_{xz}$, $M_{yz}$, center of mass of a solid $(\bar{x}, \bar{y}, \bar{z})$.

8. Cylindrical Coordinates $(r, \theta, z)$:

From CC to RC:
$$\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}$$

Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan \theta = \frac{y}{x}$ (make sure $\theta$ is in correct quadrant).

9. Spherical Coordinates $(\rho, \theta, \phi)$, where $0 \leq \phi \leq \pi$:

From SC to RC:
$$\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi
\end{align*}$$

Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan \theta = \frac{y}{x}$ and $\cos \phi = \frac{z}{\rho}$.

10. Triple integrals in Cylindrical Coordinates:
$$\begin{align*}
\iiint_E f(x, y, z) \, dV &= \iiint_E f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \\
&\uparrow
\end{align*}$$

11. Triple integrals in Spherical Coordinates:
$$\begin{align*}
\iiint_E f(x, y, z) \, dV &= \iiint_E f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&\uparrow
\end{align*}$$

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $F(x, y) = \langle P(x, y), Q(x, y) \rangle$ and $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$; $F$ is a conservative vector field if $F = \nabla f$, for some real-valued function $f$ (potential).
13. Line integral of a function $f(x,y)$ along $C$, parameterized by $x = x(t), \ y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x,y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$  

(independent of orientation of $C$, other properties and applications of line integrals of $f$)

**Remarks:**

(a) $\int_C f(x,y) \, ds$ is sometimes called the “line integral of $f$ with respect to arc length”

(b) $\int_C f(x,y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt$

(c) $\int_C f(x,y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt$

14. Line integral of vector field $F(x,y)$ along $C$, parameterized by $r(t)$ and $a \leq t \leq b$, is given by

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt.$$  

(depending on orientation of $C$, other properties and applications of line integrals of $F$)

15. Connection between line integral of vector fields and line integral of functions:

$$\int_C F \cdot dr = \int_C (F \cdot T) \, ds$$

where $T$ is the unit tangent vector to the curve $C$.

16. If $F(x,y) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j}$, then $\int_C F \cdot dr = \int_C P(x,y) \, dx + Q(x,y) \, dy$; Work $= \int_C F \cdot dr$. 