1. Vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)

(a) \( \mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \); vector addition and subtraction geometrically using parallelograms spanned by \( \mathbf{u} \) and \( \mathbf{v} \); length or magnitude of \( \mathbf{v} = \langle a, b, c \rangle \), \( |\mathbf{v}| = \sqrt{a^2 + b^2 + c^2} \); directed vector from \( P_0(x_0, y_0, z_0) \) to \( P_1(x_1, y_1, z_1) \) given by \( \mathbf{v} = \overrightarrow{P_0P_1} = P_1 - P_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \).

(b) Dot (or inner) product of \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \): \( \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \); properties of dot product; useful identity: \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \); angle between two vectors \( \mathbf{a} \) and \( \mathbf{b} \):
\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} ; \quad \mathbf{a} \perp \mathbf{b} \text{ if and only if } \mathbf{a} \cdot \mathbf{b} = 0 ; \quad \text{the vector in } \mathbb{R}^2 \text{ with length } r \text{ with angle } \theta \text{ is } \mathbf{v} = \langle r \cos \theta, r \sin \theta \rangle : \\
\]

(c) Cross product (only for vectors in \( \mathbb{R}^3 \)):
\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{vmatrix}
= \begin{vmatrix}
a_2 & a_3 \\
a_1 & a_3 \\
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
a_1 & a_3 \\
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
a_1 & a_2 \\
\end{vmatrix} \mathbf{k}
\]
properties of cross products; \( \mathbf{a} \times \mathbf{b} \) is perpendicular (orthogonal or normal) to both \( \mathbf{a} \) and \( \mathbf{b} \); area of parallelogram spanned by \( \mathbf{a} \) and \( \mathbf{b} \) is \( A = |\mathbf{a} \times \mathbf{b}| \):
\[
\]
the area of the triangle spanned is \( A = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| : \\
\]
Volume of the parallelopiped spanned by \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) is \( V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \):

2. Equation of a line \( L \) through \( P_0(x_0, y_0, z_0) \) with direction vector \( \mathbf{d} = \langle a, b, c \rangle \):

\[
\text{Vector Form: } \mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \mathbf{d}.
\]

\[
\text{Parametric Form: } \begin{cases} 
  x = x_0 + a t \\
  y = y_0 + b t \\
  z = z_0 + c t
\end{cases}
\]

\[
\text{Symmetric Form: } \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \quad (\text{If say } b = 0, \text{ then } \frac{x - x_0}{a} = \frac{z - z_0}{c}, y = y_0.)
\]

3. Equation of the plane through the point \( P_0(x_0, y_0, z_0) \) and perpendicular to the vector \( \mathbf{n} = \langle a, b, c \rangle \) (\( \mathbf{n} \) is a normal vector to the plane) is \( \langle (x - x_0), (y - y_0), (z - z_0) \rangle \cdot \mathbf{n} = 0 \); Sketching planes (consider \( x, y, z \) intercepts).

4. Quadric surfaces (can sketch them by considering various traces, i.e., curves resulting from the intersection of the surface with planes \( x = k, y = k \) and/or \( z = k \)); some generic equations have the form:

(a) Ellipsoid: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

(b) Elliptic Paraboloid: \( \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)

(c) Hyperbolic Paraboloid (Saddle): \( \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \)

(d) Cone: \( \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)

(e) Hyperboloid of One Sheet: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \)

(f) Hyperboloid of Two Sheets: \( -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)
5. Vector-valued functions $\vec{r}(t) = (f(t), g(t), h(t))$; tangent vector $\vec{r}'(t)$ for smooth curves, unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$; principal unit normal vector $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$; differentiation rules for vector functions, including:

(i) $\{\phi(t) \vec{v}(t)\}' = \phi(t) \vec{v}'(t) + \phi'(t) \vec{v}(t)$, where $\phi(t)$ is a real-valued function
(ii) $(\vec{u} \cdot \vec{v})' = \vec{u} \cdot \vec{v}' + \vec{u}' \cdot \vec{v}$
(iii) $(\vec{u} \times \vec{v})' = \vec{u} \times \vec{v}' + \vec{u}' \times \vec{v}$
(iv) $\{\vec{v}(\phi(t))\}' = \phi'(t) \vec{v}'(\phi(t))$, where $\phi(t)$ is a real-valued function

6. Integrals of vector functions $\int \vec{r}(t) \, dt = \left\langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right\rangle$; arc length of curve parameterized by $\vec{r}(t)$ is $L = \int_a^b |\vec{r}'(t)| \, dt$; arc length function $s(t) = \int_a^t |\vec{r}'(u)| \, du$; reparameterize by arc length: $\vec{\sigma}(s) = \vec{r}(t(s))$, where $t(s)$ is the inverse of the arc length function $s(t)$; the curvature of a curve parameterized by $\vec{r}(t)$ is $\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|^3}$. \textbf{Note:} $\sqrt{\alpha^2} = |\alpha|$.

7. $\vec{r}(t) = \text{position of a particle}, \vec{r}'(t) = \vec{v}(t) = \text{velocity}; \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \text{acceleration}; |\vec{r}'(t)| = |\vec{v}(t)| = \text{speed}; \text{Newton's 2}\text{nd Law:} \vec{F} = m \vec{a}$.

8. Domain and range of a function $f(x, y)$ and $f(x, y, z)$; level curves (or contour curves) of $f(x, y)$ are the curves $f(x, y) = k$; using level curves to sketch surfaces; level surfaces of $f(x, y, z)$ are the surfaces $f(x, y, z) = k$.

9. Limits of functions $f(x, y)$ and $f(x, y, z)$; limit of $f(x, y)$ does not exist if different approaches to $(a, b)$ yield different limits; continuity. \textbf{NOT REQUIRED}

10. Partial derivatives $\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$, $\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$; higher order derivatives: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$, etc; mixed partials.

11. Equation of the tangent plane to the graph of $z = f(x, y)$ at $(x_0, y_0, z_0)$ is given by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

12. Total differential for $z = f(x, y)$ is $dz = df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$; total differential for $w = f(x, y, z)$ is $dw = df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz$; linear approximation for $z = f(x, y)$ is given by $\Delta z \approx dz$, i.e., $f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$, where $\Delta x = dx, \Delta y = dy$;

\textbf{Linearization of} $f(x, y)$ at $(a, b)$ is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$; $L(x, y) \approx f(x, y)$ near $(a, b)$.
13. **CHAIN RULE**: different forms of the Chain Rule: Form 1, Form 2; **CHAIN RULE (GENERAL FORM)**: Tree diagrams. For example:

(a) If \( z = f(x, y) \) and \( \begin{cases} x = x(t) \\ y = y(t) \end{cases} \), then \( \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \):

(b) If \( z = f(x, y) \) and \( \begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases} \), then

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \]

etc.....

14. **Implicit Differentiation**:

**Part I**: If \( F(x, y) = 0 \) defines \( y \) as function of \( x \) (i.e., \( y = y(x) \)), then to compute \( \frac{dy}{dx} \), differentiate both sides of the equation \( F(x, y) = 0 \) w.r.t. \( x \) and solve for \( \frac{dy}{dx} \).

If \( F(x, y, z) = 0 \) defines \( z \) as function of \( x \) and \( y \) (i.e. \( z = z(x, y) \)), then to compute \( \frac{\partial z}{\partial x} \), differentiate the equation \( F(x, y, z) = 0 \) w.r.t. \( x \) (hold \( y \) fixed) and solve for \( \frac{\partial z}{\partial x} \). For \( \frac{\partial z}{\partial y} \), differentiate the equation \( F(x, y, z) = 0 \) w.r.t. \( y \) (hold \( x \) fixed) and solve for \( \frac{\partial z}{\partial y} \).

**Part II**: If \( F(x, y) = 0 \) defines \( y \) as function of \( x \) \( \Rightarrow \) \( \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} \);

while if \( F(x, y, z) = 0 \) defines \( z \) as function of \( x \) and \( y \) \( \Rightarrow \) \( \frac{\partial z}{\partial x} = -\frac{\partial F/\partial z}{\partial F/\partial x} \) and \( \frac{\partial z}{\partial y} = -\frac{\partial F/\partial z}{\partial F/\partial y} \).
15. Gradient vector for $f(x, y)$: $\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$, properties of gradients; gradient points in direction of maximum rate of increase of $f$, maximum rate of increase is $|\nabla f|$; $\nabla f(x_0, y_0) \perp$ level curve $f(x, y) = k$ and, in the case of 3 variables, $\nabla f(x_0, y_0, z_0) \perp$ level surface $f(x, y, z) = k$.

16. Directional derivative of $f(x, y)$ at $(x_0, y_0)$ in the direction $\vec{u}$: $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$, where $\vec{u}$ must be a unit vector; tangent planes to level surfaces $f(x, y, z) = k$ (a normal vector at $(x_0, y_0, z_0)$ is $\vec{n} = \nabla f(x_0, y_0, z_0)$).

17. Relative/local extrema; critical points (points where $\nabla f = \vec{0}$ or $\nabla f$ does not exist).

18. 2\textsuperscript{nd} Derivatives Test: Suppose the 2\textsuperscript{nd} partials of $f(x, y)$ are continuous in a disk with center $(a, b)$ and $\nabla f(a, b) = \vec{0}$. Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a, b)}$.

(a) If $D > 0$ and $f_{xx}(a, b) > 0 \implies f(a, b)$ is a local minimum value.
(b) If $D > 0$ and $f_{xx}(a, b) < 0 \implies f(a, b)$ is a local maximum value.
(c) If $D < 0 \implies f(a, b)$ is a not a local min or local max value. So $(a, b)$ is a saddle point of $f$.

If $D = 0$ (or if $\nabla f(a, b)$ does not exist or $f$ has more than 2 variables) the test gives no information and you need to do something else to determine if a relative extremum exists.