Study Guide for Exam 2

1. You are supposed to know how to carry out the integration by parts
   • Indefinite form \( \int u dv = uv - \int v du \)
   • Definite form \( \int_a^b u dv = [uv]_a^b - \int_a^b v du \)

Example Problems
1.1. Compute the following integrals:
   (i) \( \int x e^x dx \)
   (ii) \( \int e^x \sin x dx \)
   (iii) \( \int \ln x dx \)
   (iv) \( \int \sin^{-1} x dx \)

1.2. Evaluate the following integrals
   (i) \( \int_0^{\pi/2} x \cos(2x) dx \)
   (ii) \( \int_0^{1/2} \sin^{-1} x dx \)
   (iii) \( \int_0^1 \tan^{-1} x dx \)

1.3. Find the volume of the solid obtained by rotating the following region about the y-axis: the region is bounded by \( y = f(x) = \sin x \) and the x-axis on the interval \([0, \pi] \).

2. You are supposed to know how to compute the integration of the form
   (1) \( \int \sin^m x \cos^n x \ dx \)
      • Case: \( m \) odd \( \rightarrow \) Use \( u = \cos x \) substitution
      • Case: \( n \) odd \( \rightarrow \) Use \( u = \sin x \) substitution
      • Case: \( m \) & \( n \) even \( \rightarrow \) Reduce the degree by double angle formula
   (2) \( \int \tan^m x \sec^n x \ dx \)
      • Case: \( n > 0 \) even \( \rightarrow \) Use \( u = \tan x \) substitution
      • Case: \( n > 0 \) odd & \( m \) odd \( \rightarrow \) Use \( u = \sec x \) substitution
      • Case: \( n > 0 \) & \( m \) even \( \rightarrow \) Integration by parts
      • Case: \( n = 0 \) \( \rightarrow \) Use \( \tan^2 x = \sec^2 x - 1 \) to reduce to the case \( n > 0 \) and to the lower degree case
Example Problems

2.1. Compute the following integrals:

(i) \( \int \sin^3 x \cos^2 x \, dx \)

(ii) \( \int \sin^2 x \cos^3 x \, dx \)

(iii) \( \int \sin^3 x \cos^3 x \, dx \)

(iv) \( \int \sin^4 x \cos^2 x \, dx \)

2.2. Compute the following integrals:

(i) \( \int \tan^2 x \sec^4 x \, dx \)

(ii) \( \int \sec^3 \tan x \, dx \)

(iii) \( \int \sec^3 x \, dx \).

(iv) \( \int \tan^2 x \sec x \, dx \).

(v) \( \int \tan^5 x \, dx \).

2.3.

(i) Compute \( \int \sec x \, dx = \int \frac{1}{\cos^2 x} \cos x \, dx = \int \frac{1}{1 - \sin^2 x} \cos x \, dx \)
using the substitution \( u = \sin x \) and then using the partial fraction.

(ii) Check that the result obtained in (i) coincides with the well-known formula \( \int \sec x \, dx = \ln |\sec x + \tan x| + C \).

2.4.

We would like to compute

\( \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \)

in the following two ways.

(i) Use substitution \( u = \sin x \) to get

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{\sin x}{\cos^2 x} \, dx = \int \frac{\sin x}{1 - \sin^2 x} \, dx = \int \frac{u}{1 - u^2} \, du
\]

and then use the partial fractions.
(ii) Use the same substitution as above and then use another substitution $v = 1 - u^2$ to get
\[ \int \tan x \, dx = \int \frac{u}{1 - u^2} \, du = \int -\frac{1}{2} \, dv \]
and compute.

(iii) Check that the results obtained in (i) and (ii) coincide with the well known formula $\int \tan x \, dx = \ln |\sec x| + C$.

3. You are supposed to know how to use the 3 types of trigonometric substitution, and carry out the integration accordingly.

1. $\sqrt{a^2 - x^2}, x = a \sin \theta, dx = a \cos \theta, \sqrt{a^2 - x^2} = a \cos \theta$,
2. $\sqrt{a^2 + x^2}, x = a \tan \theta, dx = a \sec^2 \theta, \sqrt{a^2 + x^2} = a \sec \theta$,
3. $\sqrt{x^2 - a^2}, x = a \sec \theta, dx = a \tan \theta \sec \theta, \sqrt{x^2 - a^2} = a \tan \theta$.

**Example Problems**

3.1. Compute the following integrals:

(i) $\int \frac{dx}{\sqrt{4 - x^2}}$

(ii) $\int \frac{\sqrt{5 - 4x^2}}{dx}$

(iii) $\int_0^6 \frac{\sqrt{x^2 - 9}}{x} \, dx$

(iv) $\int \frac{\sqrt{x^2 + 1}}{dx}$

(v) $\int \frac{\sqrt{3}}{x^2 \sqrt{x^2 + 1}} \, dx$

(vi) $\int \frac{\sqrt{3 - 2x - x^2}}{x^2 - 2x + 2} \, dx$

(vii) $\int \frac{\sqrt{x^2 - 2x + 10}}{x} \, dx$

(viii) $\int \frac{x}{\sqrt{3 + 2x - x^2}} \, dx$

3.2. Verify that the area of a circle of radius $r$ is $\pi r^2$.

4. You are supposed to know
   - the proper form of the partial fractions,
   - how to determine the appropriate constants appearing in the partial fraction,
   - how to compute the integral accordingly.
Example Problems

4.1. Determine the proper form of the partial fractions for the following. (You do not have to calculate the constants.)

(i) \( \frac{1}{(x+2)(x^2-4)((x^2+x+1)^2} \)

(ii) \( \frac{1}{(x-1)(x^3-1)(x^2+4x+5)} \)

4.2. Compute the following integrals:

(i) \( \int \frac{x^2}{(x-1)^2} \, dx \)

(ii) \( \int \frac{x + 2}{x^2 + 2x + 2} \, dx \)

(iii) \( \int \frac{x}{(x + 1)(x - 1)(x - 2)} \, dx \)

(iv) \( \int \frac{x^2 + 2x + 2}{x^2 + 4x + 5} \, dx \)

(v) \( \int \frac{x^2 + x + 2}{x^2 + 4x + 5} \, dx \)

5. You are supposed to know why a given improper integral is improper, and accordingly to be able to determine if the given improper integral is convergent/divergent. In case it is convergent, you should be able to compute its value.

Example Problems

5.1. Evaluate the following improper integrals

(i) \( \int_0^{\infty} \frac{e^x}{e^{2x} + 1} \, dx \)

(ii) \( \int_0^3 \frac{e^x}{e^{2x} + 1} \, dx \)

(iii) \( \int_0^9 \frac{1}{x - 1} \, dx \)

(iv) \( \int_0^9 \frac{1}{\sqrt{x - 1}} \, dx \)

(v) \( \int_{-\infty}^{\infty} x \, dx \)

(vi) \( \int_0^{\infty} xe^{-x} \, dx \)

(vi) \( \int_{-\infty}^{\infty} xe^{-x^2} \, dx \)
6. You are supposed to be able to determine if a given sequence is convergent/divergent. In case it is convergent, you should be able to compute its limit.

**Example Problems**

6.1. Compute the limit $\lim_{n \to \infty} a_n$ of the following sequences:

(i) $a_n = \frac{(-1)^n n}{n^2 + 1}$

(ii) $a_n = \tan^{-1} \left( \frac{n^3 + 5}{n^2} \right)$

(iii) $a_n = (-1)^n \sin \left( \frac{\pi}{2} - \frac{1}{n} \right)$

(iv) $a_n = \cos(n\pi)$

(v) $a_n = \sin(n\pi)$

(vi) $a_n = \frac{n!n^2}{(n + 2)!} \cos(1/2n)$

(vii) $a_n = n \tan(1/n)$