Study Guide for Exam 2

1. You are supposed to know how to carry out the integration by parts
   - Indefinite form $\int udv = uv - \int vdu$
   - Definite form $\int_a^b udv = [uv]_a^b - \int_a^b vdu$

Example Problems

1.1. Compute the following integrals:

$$\text{(i)} \quad \int xe^x dx$$
$$\text{(ii)} \quad \int e^x \sin x dx$$
$$\text{(iii)} \quad \int \ln x \; dx$$
$$\text{(iv)} \quad \int \sin^{-1} x \; dx$$

1.2. Evaluate the following integrals

$$\text{(i)} \quad \int_0^{\frac{\pi}{6}} x\cos(2x) \; dx$$
$$\text{(ii)} \quad \int_0^{1/2} \sin^{-1} x \; dx$$
$$\text{(iii)} \quad \int_0^1 \tan^{-1} x \; dx$$

1.3. Find the volume of the solid obtained by rotating the follwoing region about the y-axis: the region is bounded by $y = f(x) = \sin x$ and the x-axis on the interval $[0, \pi]$.

2. You are supposed to know how to comopute the integration of the form

(1) $\int \sin^m x \cos^n x \; dx$

- Case: m odd $\rightarrow$ Use $u = \cos x$ substitution
- Case: n odd $\rightarrow$ Use $u = \sin x$ substitution
- Case: m & n even $\rightarrow$ Reduce the degree by double angle formula

(2) $\int \tan^m x \sec^n x \; dx$

- Case: $n > 0$ even $\rightarrow$ Use $u = \tan x$ substitution
- Case: $n > 0$ odd & m odd $\rightarrow$ Use $u = \sec x$ substitution
- Case: $n > 0$ & m even $\rightarrow$ Integration by parts
- Case: $n = 0 \rightarrow$ Use $\tan^2 x = \sec^2 x - 1$ to reduce to the case $n > 0$ and to the lower degree case

Example Problems

2.1. Compute the following integrals:

$$\text{(i)} \quad \int \sin^3 x \cos^2 x \; dx$$
$$\text{(ii)} \quad \int \sin^2 x \cos^3 x \; dx$$
$$\text{(iii)} \quad \int \sin^3 x \cos^3 x \; dx$$
$$\text{(iv)} \quad \int \sin^4 x \cos^2 x \; dx$$

2.2. Compute the following integrals:

$$\text{(i)} \quad \int \tan^2 x \sec^4 x \; dx$$
$$\text{(ii)} \quad \int \sec^3 \tan x \; dx$$
$$\text{(iii)} \quad \int \sec^3 x \; dx.$$
$$\text{(iv)} \quad \int \tan^2 x \sec x \; dx.$$
$$\text{(v)} \quad \int \tan^5 x \; dx.$$

2.3.

(i) Compute $\displaystyle\int \sec x \; dx = \int \frac{1}{\cos^2 x} \cos x \; dx = \int \frac{1}{1 - \sin^2 x} \cos x \; dx$ using the substitution $u = \sin x$ and then using the partial fraction.

(ii) Check that the result obtained in (i) coincides with the well-known formula $\int \sec x \; dx = \ln|\sec x + \tan x| + C$.

2.4.

We would like to compute

$$\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$$

in the following two ways.

(i) Use substitution $u = \sin x$ to get

$$\begin{aligned} \int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx = \int \frac{\sin x}{\cos^2 x} \; \cos x dx \\ &= \int \frac{\sin x}{1 - \sin^2 x} \; \cos x dx = \int \frac{u}{1 - u^2} \; du \end{aligned}$$

and then use the partial fractions.

(ii) Use the same substitution as above and then use another substitution $v = 1 - u^2$ to get

$$\int \tan x \; dx = \int \frac{u}{1-u^2} \; du = \int -\frac{1}{2}\frac{dv}{v}$$

and compute.

(iii) Check that the results obtained in (i) and (ii) coincide with the well known formula $\int \tan x \; dx = \ln|\sec x| + C$.

3. You are supposed to know how to use the 3 types of trigonometric substitution, and carry out the integration accordingly.

(1) $\sqrt{a^2 - x^2}, x = a\sin\theta, dx = a\cos\theta, \sqrt{a^2 - x^2} = a\cos\theta,$

(2) $\sqrt{a^2 + x^2}, x = a\tan\theta, dx = a\sec^2\theta, \sqrt{a^2 + x^2} = a\sec\theta,$

(3) $\sqrt{x^2 - a^2}, x = a\sec\theta, dx = a\tan\theta\sec\theta, \sqrt{x^2 - a^2} = a\tan\theta.$

Example Problems

3.1. Compute the following integrals:

$$
\begin{array}{ll}
\text{(i)} & \displaystyle\int \frac{dx}{\sqrt{4-x^2}} \\
\text{(ii)} & \displaystyle\int \sqrt{5-4x^2} \; dx \\
\text{(iii)} & \displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x} \; dx \\
\text{(iv)} & \displaystyle\int \sqrt{x^2+1} \; dx \\
\text{(v)} & \displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+1}} \\
\text{(vi)} & \displaystyle\int \frac{x}{\sqrt{3-2x-x^2}} dx \\
\text{(vii)} & \displaystyle\int \frac{x^2-2x+2}{\sqrt{x^2-2x+10}} dx \\
\text{(viii)} & \displaystyle\int \frac{x}{\sqrt{3+2x-x^2}} dx
\end{array}
$$

3.2. Verify that the area of a circle of radius r is πr^2.

4. You are supposed to know

∘ the proper form of the partial fractions,

∘ how to determine the appropariate constants apperaring in the partial fraction,

∘ how to compute the integral accordingly.

Example Problems

4.1. Determine the proper form of th partial fractions for the following. (You do not have to calculate the constants.)

$$\text{(i)} \quad \frac{1}{(x+2)(x^2-4)((x^2+x+1)^2}$$

$$\text{(ii)} \quad \frac{x^3}{(x-1)(x^3-1)(x^2+4x+5)}$$

4.2. Compute the following integrals:

$$\text{(i)} \quad \int \frac{x^2}{(x-1)^2}\, dx$$

$$\text{(ii)} \quad \int \frac{x+2}{x^2+2x+2}\, dx$$

$$\text{(iii)} \quad \int \frac{x}{(x+1)(x-1)(x-2)}\, dx$$

$$\text{(iv)} \quad \int \frac{x^2}{(x-1)^2(x^2+1)}\, dx$$

$$\text{(v)} \quad \int \frac{x^2+x+2}{x^2+4x+5}\, dx$$

5. You are supposed to know why a given improper integral is improper, and accordingly to be able to determine if the given improper integral is convergent/divergent. In case it is convergent, you should be able to compute its value.

Example Problems

5.1. Evaluate the following improper integrals

$$\text{(i)} \quad \int_0^\infty \frac{e^x}{e^{2x}+1}\, dx$$

$$\text{(ii)} \quad \int_0^\infty \frac{e^{2x}}{e^{2x}+1}\, dx$$

$$\text{(iii)} \quad \int_0^9 \frac{1}{x-1}\, dx$$

$$\text{(iv)} \quad \int_0^9 \frac{1}{\sqrt[3]{x-1}}\, dx$$

$$\text{(v)} \quad \int_{-\infty}^\infty x\, dx$$

$$\text{(vi)} \quad \int_0^\infty xe^{-x}\, dx$$

$$\text{(vi)} \quad \int_{-\infty}^\infty xe^{-x^2}\, dx$$

6. You are supposed to be able to determine if a given sequence is convergent/divergent. In case it is convergent, you should be able to compute its limit.

Example Problems

6.1. Compute the limit $\lim_{n\to\infty} a_n$ of the following sequences:

$$
\begin{array}{ll}
\text{(i)} & a_n = \dfrac{(-1)^n n}{n^2+1} \\
\text{(ii)} & a_n = \tan^{-1}\left(\dfrac{n^3+5}{n^2}\right) \\
\text{(iii)} & a_n = (-1)^n \sin\left(\dfrac{\pi}{2} - \dfrac{1}{n}\right) \\
\text{(iv)} & a_n = \cos(n\pi) \\
\text{(v)} & a_n = \sin(n\pi) \\
\text{(vi)} & a_n = \dfrac{n!n^2}{(n+2)!}\cos(1/2n) \\
\text{(vii)} & a_n = n\tan(1/n)
\end{array}
$$