Answer Keys

for the example problems

in Study Guide for Exam 2.

1.1.

(i) $\int x e^x \, dx = \int u \, dv$

$$\begin{pmatrix} u = x & v = e^x \\ du = dx & dv = e^x dx \end{pmatrix}$$

$= uv - \int v \, du$

$= x e^x - \int e^x \, dx$

$= x e^x - e^x + C$

① $\int e^x \sin x \, dx = \int u \, dv$ ②

$$\left(\begin{array}{ll} u = e^x & v = (-\cos x) \\ du = e^x dx & dv = \sin x \, dx \end{array} \right)$$

$= uv - \int v \, du$

$= e^x(-\cos x) - \int (-\cos x) e^x dx$

$= e^x(-\cos x) + \int e^x \cos x \, dx$

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② $\int e^x \cos x \, dx = \int u \, dv$

$$\left(\begin{array}{ll} u = e^x & v = \sin x \\ du = e^x dx & dv = \cos x \, dx \end{array} \right)$$

$= uv - \int v \, du$

$= e^x \sin x - \int \sin x \cdot e^x dx$

$= e^x \sin x - \int e^x \sin x \, dx$

Plug the result of ② into ①: (3)

$$\int e^x \sin x \, dx = e^x(-\cos x) + \int e^x \cos x \, dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

We conclude

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C$$

$\rightarrow$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x(-\cos x + \sin x) + C$$

(iii) $\int \ln x \, dx = \int u \, dv$ 4

$$\left(\begin{array}{ll} u = \ln x & v = x \\ du = \frac{1}{x} dx & dv = dx \end{array} \right)$$

$$= u \cdot v - \int v \, du$$

$$= \ln x \cdot x - \int \underbrace{x \cdot \frac{1}{x}}_{= 1} dx$$

$$= x \cdot \ln x - x + C.$$

(iv) $\int \sin^{-1}x \, dx = \int u \, dv$ 5

①

$$\left(\begin{array}{ll} u = \sin^{-1}x & v = x \\ du = \dfrac{1}{\sqrt{1-x^2}}\,dx & dv = dx \end{array} \right)$$

$$= uv - \int v \, du$$

$$= \sin^{-1}x \cdot x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

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② $\displaystyle \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}}$

$$\left(\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array} \right)$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\cdot 2\sqrt{u} + C$$

$$= -\sqrt{u} + C = -\sqrt{1-x^2} + C$$

Plug the result of ② into ①: (6)

$$\int \sin^{-1} x \, dx$$

$$= \sin^{-1} x \cdot x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= \sin^{-1} x \cdot x - \left(-\sqrt{1 \underset{-}{\cancel{+}} x^2} + \underset{C_{old}}{\underset{\shortparallel}{C}} \right)$$

$$= x \cdot \sin^{-1} x + \sqrt{1 \underset{-}{\cancel{+}} x^2} + \underset{C_{new}}{\underset{\shortparallel}{C}}$$

1.2. 7

(i) $$\int_0^{\pi/6} x \cos(2x)\, dx = \int_0^{\pi/6} u\, dv$$

$$\left(\begin{array}{ll} u = x & v = \frac{1}{2}\sin(2x) \\ du = dx & dv = \cos(2x)\, dx \end{array} \right)$$

$$= [uv]_0^{\pi/6} - \int_0^{\pi/6} v\, du$$

$$= \left[x \cdot \frac{1}{2}\sin(2x) \right]_0^{\pi/6} - \int_0^{\pi/6} \frac{1}{2}\sin(2x)\, dx$$

$$= \left[x \cdot \frac{1}{2}\sin(2x) \right]_0^{\pi/6} - \left[\frac{1}{2}\left(-\frac{1}{2}\cos 2x \right) \right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[x \cdot \sin(2x) \right]_0^{\pi/6} + \frac{1}{4}\left[\cos(2x) \right]_0^{\pi/6}$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - 0 \right] + \frac{1}{4}\left[\frac{1}{2} - 1 \right]$$

$$= \frac{\pi}{24}\sqrt{3} - \frac{1}{8}$$

8

(ii) (1) $\int_0^{1/2} \sin^{-1}x \, dx = \int_0^{1/2} u \, dv$ (9)

$$\left(\begin{array}{ll} u = \sin^{-1}x & v = x \\ du = \frac{1}{\sqrt{1-x^2}} dx & dv = dx \end{array} \right)$$

$$= \left[\sin^{-1}x \cdot x \right]_0^{1/2} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

$$\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx = \int_1^{3/4} \frac{-\frac{1}{2} du}{\sqrt{u}}$$

$$\left(\begin{array}{lll} x & u = 1-x^2 & du = -2x\,dx \\ 1/2 & 3/4 & \\ 0 & 1 & \end{array} \right)$$

$$= -\frac{1}{2} \int_1^{3/4} \frac{1}{\sqrt{u}} du = -\frac{1}{2} \left[2\sqrt{u} \right]_1^{3/4}$$

$$= -\frac{\sqrt{3}}{2} + 1$$

Plug the result of ② into ①:

$$\int_0^{1/2} \sin^{-1} x \, dx$$

$$= \left[\sin^{-1} \cdot x \right]_0^{1/2} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= \frac{\pi}{6} \cdot \frac{1}{2} - \left[-\frac{\sqrt{3}}{2} + 1 \right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

(iii)

① $$\int_0^1 \tan^{-1}x\,dx = \int_0^1 u\,dv$$

$$\left(\begin{array}{ll} u = \tan^{-1}x & v = x \\ du = \dfrac{1}{1+x^2}\,dx & dv = dx \end{array}\right)$$

$$= [uv]_0^1 - \int_0^1 v\,du$$

$$= [\tan^{-1}x \cdot x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

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② $$\int_0^1 \frac{x}{1+x^2}\,dx$$

$$\left(\begin{array}{lll} x & u = 1+x^2 & du = 2x\,dx \\ 1 & 2 & \\ 0 & 1 & \end{array}\right)$$

$$= \int_1^2 \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}[\ln|u|]_1^2 = \frac{1}{2}\ln 2$$

Plug the results of ② into ①:

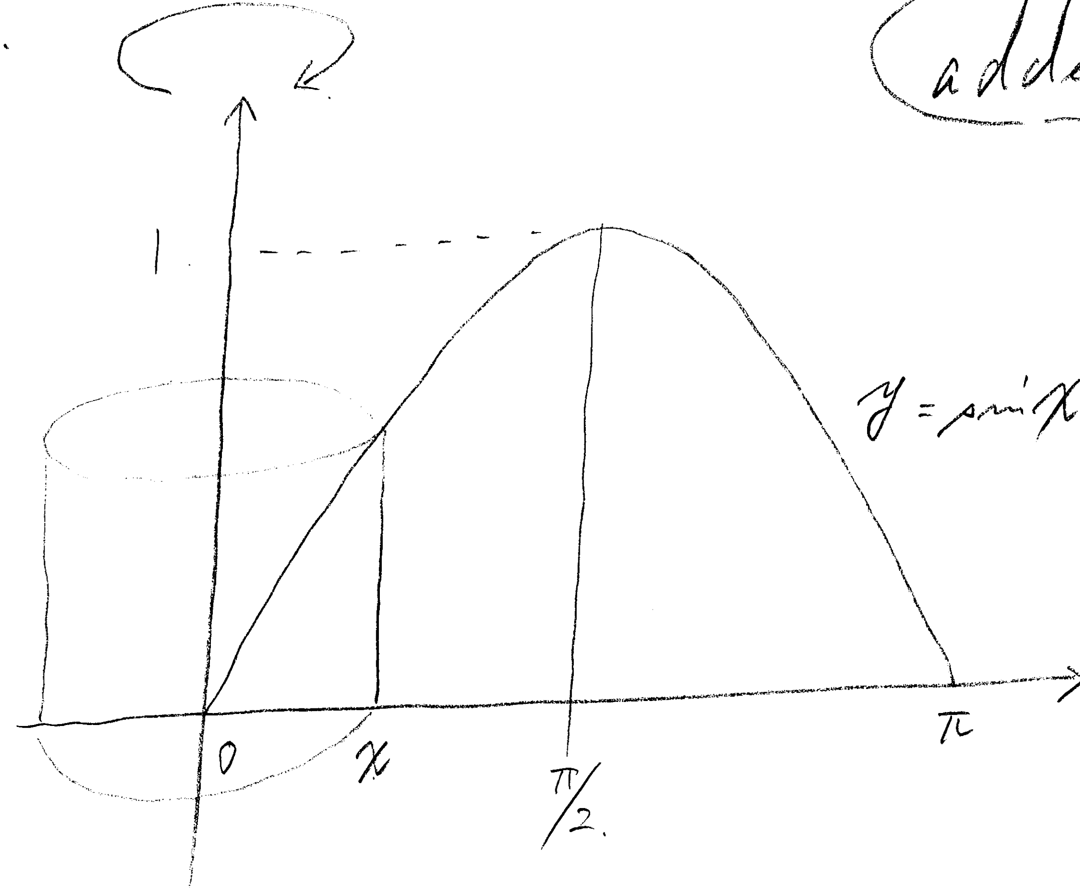
$$\int_0^1 \tan^{-1} x \, dx$$

$$= \left[\tan^{-1} x \cdot x \right]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

1.3.

added



By the method of cylindrical shell, we have

$$V = \int_0^{\pi} 2\pi x \cdot \sin x$$

$$= 2\pi \int_0^{\pi} x \sin x \, dx = 2\pi^2$$

$$\int_0^{\pi} x \sin x \, dx$$

added

$$\left(\begin{array}{ll} u = x & v = -\cos x \\ du = dx & dv = \sin x \, dx \end{array} \right)$$

$$= \int_0^{\pi} u \, dv$$

$$= [uv]_0^{\pi} - \int_0^{\pi} v \, du$$

$$= \underbrace{[x \cdot (-\cos x)]_0^{\pi}}_{= \pi} - \underbrace{\int_0^{\pi} (-\cos x) \, dx}_{+ \underbrace{\int_0^{\pi} \cos x \, dx}_{= [\sin x]_0^{\pi}}}$$

2.1.

(i) $\int \sin^3 x \cos^2 x \, dx$

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$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$\left(\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned} \right)$$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$= \int (1 - u^2) u^2 (-du)$$

$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

(ii) $\int \sin^2 x \cos^3 x \, dx$

(14)

$= \int \sin^2 x \cos^2 x \cos x \, dx$

$\left(\begin{array}{l} u = \sin x \\ du = \cos x \end{array} \right)$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$= \int u^2 (1 - u^2) \, du$

$= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(iii)

① $$\int \sin^3 x \cos^3 x \, dx$$

15

$$= \int \sin^2 x \cos^3 x \sin x \, dx$$

$$\left(\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right)$$

$$= \int (1 - \cos^2 x) \cos^3 x \cdot \sin x \, dx$$

$$= \int (1 - u^2) u^3 (-du)$$

$$= \int (u^5 - u^3) \, du = \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C.$$

(2) $\int \sin^3 x \cos^3 x \, dx$

16

$$= \int \sin^3 x \cos^2 x \cos x \, dx$$

$$\left(\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right)$$

$$= \int \sin^3 x \, (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^3 (1 - u^2) \, du$$

$$= \int (u^3 - u^5) \, du = \frac{u^4}{4} - \frac{u^6}{6} + C$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

(iv)

① $\int \sin^4 x \cos^2 x \, dx$ 17

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right) dx$$

$$= \frac{1}{8} \int \{1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\} \, dx$$

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② $\int \cos(2x) \, dx = \frac{1}{2} \sin(2x) + C$

$$\int \cos^2(2x) \, dx = \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{1}{2} x + \frac{1}{8} \sin(4x) \, dx$$

$$\int \cos^3(2x)\,dx$$

(18)

$$= \int \cos^2(2x)\cos(2x)\,dx$$

$$\left(\begin{array}{l} u = \sin(2x) \\ du = 2\cos(2x) \end{array}\right)$$

$$= \int \{1 - \sin^2(2x)\}\cos(2x)\,dx$$

$$= \int \{1 - u^2\}\frac{1}{2}\,du$$

$$= \frac{1}{2}u - \frac{1}{6}u^3 + C$$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C$$

Plug the results of ② into ①:

$$\int \sin^4 x \cos^2 x \, dx$$

$$= \frac{1}{8} \int \{ 1 - \cos(2x) - \cos^2(2x) + \cos^3(2x) \} \, dx$$

$$= \frac{1}{8} \{ x$$

$$- \frac{1}{2} \sin(2x)$$

$$- \{ \frac{1}{2} x + \frac{1}{8} \sin(4x) \}$$

$$+ \frac{1}{2} \sin(2x) - \frac{1}{6} \sin^3(2x) \} + C$$

$$= \frac{1}{16} x - \frac{1}{64} \sin(4x) - \frac{1}{48} \sin^3(2x)$$

$$+ C.$$

2.2

(i) $\int \tan^2 x \sec^4 x \, dx$

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$\left(\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right)$

$= \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^2 x \left(1 + \tan^2 x \right) \sec^2 x \, dx$

$= \int u^2 \left(1 + u^2 \right) du$

$= \int \left(u^2 + u^4 \right) du$

$= \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

(ii) $\displaystyle\int \sec^3 x \tan x \, dx$

21

$$\left(\begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right)$$

$$= \int \sec^2 x \cdot \sec x \tan x \, dx$$

$$= \int u^2 \cdot du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\sec^3 x}{3} + C$$

(iii) $\displaystyle\int \sec^3 x \, dx = \int u \, dv$ 22

$$\left(\begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x \, dx & dv = \sec^2 x \, dx \end{array} \right)$$

$$= uv - \int v \, du$$

$$= \sec x \tan x - \underbrace{\int \tan x \cdot \sec x \tan x \, dx}$$

$$\underbrace{\int \sec x \tan^2 x \, dx}_{\shortparallel}$$

$$\int \sec x \, (\sec^2 x - 1) \, dx$$

$$= \sec x \cdot \tan x + \int \sec x \, dx$$

$$- \int \sec^3 x \, dx$$

$\longrightarrow$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

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$\longrightarrow$

$$\int \sec^3 x\,dx = \frac{1}{2}\left\{\sec x \tan x + \ln|\sec x + \tan x|\right\} + C.$$

Note : $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

(iv) $$\int \tan^2 x \sec x \, dx = \int u \, dv$$ 24

$$\left(\begin{array}{ll} u = \tan x & v = \sec x \\ du = \sec^2 x \, dx & dv = \tan x \sec x \, dx \end{array} \right)$$

$$= uv - \int v \, du$$

$$= \tan x \cdot \sec x - \underbrace{\int \sec x \cdot \sec^2 x \, dx}$$

$$\int \sec x \left(1 + \tan^2 x \right) dx$$

$$= \tan x \sec x - \int \sec x \, dx$$

$$- \int \tan^2 x \cdot \sec x \, dx$$

$\longrightarrow$

$$2 \int \tan^2 x \sec x \, dx = \tan x \sec x - \int \sec x \, dx$$

$\longrightarrow$

25

$$\int \tan^2 x \sec x \, dx$$

$$= \frac{1}{2} \left\{ \tan x \sec x - \ln|\sec x + \tan x| \right\}$$

$$+ C$$

(V) $\displaystyle\int \tan^5 x \, dx$

26

1

$$= \int \tan^3 x \cdot \tan^2 x \, dx$$

$$= \int \tan^3 x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

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2 $\displaystyle\int \tan^3 x \sec^2 x \, dx$

$$\begin{pmatrix} u = \tan x \\ du = \sec^2 x \end{pmatrix}$$

$$= \int u^3 \, du = \frac{u^4}{4} + C$$

$$= \frac{\tan^4 x}{4} + C$$

(3)

$$\int \tan^3 x \, dx$$

27

$$= \int \tan x \cdot \tan^2 x \, dx$$

$$= \int \tan x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$\begin{pmatrix} u = \tan x \\ du = \sec^2 x \end{pmatrix} \qquad \begin{pmatrix} u = \cos x \\ du = -\sin x \, dx \end{pmatrix}$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Plug the results of 2 & 3 into 1 :

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$$\int \tan^5 x \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x|$$

$$+ C$$

2.3.

(i) $\displaystyle\int \sec x\, dx = \int \frac{1}{\cos^2 x} \cos x\, dx$ (29)

$$\left(\begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned} \right)$$

$$= \int \frac{1}{1-\sin^2 x} \cos x\, dx$$

$$= \int \frac{1}{1-u^2}\, du$$

$$= \int \frac{-1}{u^2-1}\, du$$

$$= \int \frac{1}{2}\left(\frac{1}{u+1} - \frac{1}{u-1}\right) dx$$

$$= \frac{1}{2}\left\{ \ln|u+1| - \ln|u-1| \right\} + C$$

$$= \frac{1}{2}\left\{ \ln|\sin x+1| - \ln|\sin x-1| \right\} + C.$$

2. 4

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(i) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$$= \int \frac{\sin x}{\cos^2 x} \cos x \, dx$$

$$= \int \frac{\sin x}{1 - \sin^2 x} \cos x \, dx$$

$$\left(\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned} \right)$$

$$= \int \frac{u}{1 - u^2} du$$

$$= \int \frac{-u}{u^2 - 1} du$$

$$= \int \left\{ \frac{-1/2}{u+1} + \frac{-1/2}{u-1} \right\} du$$

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$$= -\frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C$$

$$= -\frac{1}{2}\ln|\sin x+1| - \frac{1}{2}|\sin x-1| + C$$

(iii)

$$\longrightarrow$$

$$= \ln\left|\frac{1}{\sin x+1}\right|^{\frac{1}{2}} + \ln\left|\frac{1}{\sin x-1}\right|^{\frac{1}{2}} + C$$

$$= \ln\left|\frac{1}{(\sin x+1)(\sin x-1)}\right|^{\frac{1}{2}} + C$$

$$= \ln\left|\frac{1}{\sin^2 x-1}\right|^{\frac{1}{2}} + C$$

$$= \ln\left|\frac{1}{-\cos^2 x}\right|^{\frac{1}{2}} + C$$

$$= \ln\left|\frac{1}{\cos^2 x}\right|^{\frac{1}{2}} + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \ln|\sec x| + C$$

(ii) $$\int \tan x \, dx = \int \frac{u}{1-u^2}\, du$$ (32)

$$= \int -\frac{1}{2} \frac{dv}{v}$$

$$\left(\begin{array}{l} v = 1-u^2 \\ dv = -2u\, du \end{array} \right)$$

$$= -\frac{1}{2} \ln |v| + C$$

$$= -\frac{1}{2} \ln |1-u^2| + C$$

$$= -\frac{1}{2} \ln |1-\sin^2 x| + C$$

$\xrightarrow{(iii)}$

$$= -\frac{1}{2} \ln |\cos^2 x| + C$$

$$= \ln |\cos^2 x|^{-\frac{1}{2}} + C$$

$$= \ln \left|\frac{1}{\cos^2 x}\right|^{\frac{1}{2}} + C$$

$$= \ln \left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

3.1.

(i) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

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$$\left(\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \\ \sqrt{4-x^2} &= 2\cos\theta \end{aligned}\right)$$

$$= \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta$$

$$= \theta + C$$

$$\left(\begin{aligned} \frac{x}{2} &= \sin\theta \\ \sin^{-1}\left(\frac{x}{2}\right) &= \theta \end{aligned}\right)$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

(ii) $\displaystyle\int \sqrt{5-4x^2}\, dx$

(34)

$$\begin{pmatrix} u = 2x \\ du = 2\, dx \end{pmatrix}$$

$$= \int \sqrt{5-u^2} \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2} \int \sqrt{5-u^2}\, du$$

$$\begin{pmatrix} u = \sqrt{5} \sin\theta \\ du = \sqrt{5} \cos\theta\, d\theta \\ \sqrt{5-u^2} = \sqrt{5} \cos\theta \end{pmatrix}$$

$$= \frac{1}{2} \int \sqrt{5} \cos\theta \cdot \sqrt{5} \cos\theta\, d\theta$$

$$= \frac{5}{2} \int \cos^2\theta\, d\theta$$

$$= \frac{5}{2} \int \frac{1+\cos(2\theta)}{2}\, d\theta$$

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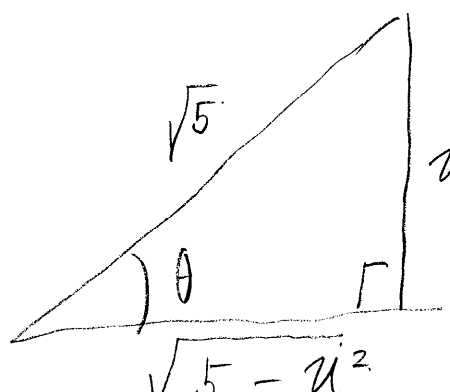
$$= \frac{5}{4} \int \{ 1 + \cos(2\theta) \} \, d\theta$$

$$= \frac{5}{4} \{ \theta + \underbrace{\frac{1}{2} \sin(2\theta)}_{\frac{1}{2} \, 2 \sin\theta \cos\theta} \} + C$$

$$= \frac{5}{4} \{ \theta + \sin\theta \cos\theta \} + C$$

$$\frac{u}{\sqrt{5}} = \sin\theta$$



$$= \frac{5}{4} \{ \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + \frac{u}{\sqrt{5}} \cdot \frac{\sqrt{5 - u^2}}{\sqrt{5}} \} + C$$

$$= \frac{5}{4} \sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + \frac{1}{4}(2x)\sqrt{5 - 4x^2} + C$$

(iii) $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx$ 36

$x = 3\sec\theta$	$\sec\theta$	θ
6	2	$\pi/3$
3	1	0

$dx = 3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9} = 3\tan\theta$

$$= \int_0^{\pi/3} \frac{3\tan\theta}{\cancel{3\sec\theta}}\,\cancel{3\sec\theta}\tan\theta\,d\theta$$

$$= 3\int_0^{\pi/3} \tan^2\theta\,d\theta$$

$$= 3\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta$$

$$= 3\left[\tan\theta - \theta\right]_0^{\pi/3}$$

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$$= 3\left[\left(\sqrt{3} - \pi/3\right) - (0 - 0)\right]$$

$$= 3\sqrt{3} - \pi.$$

(iv) $$\int \sqrt{x^2+1}\, dx$$ 38

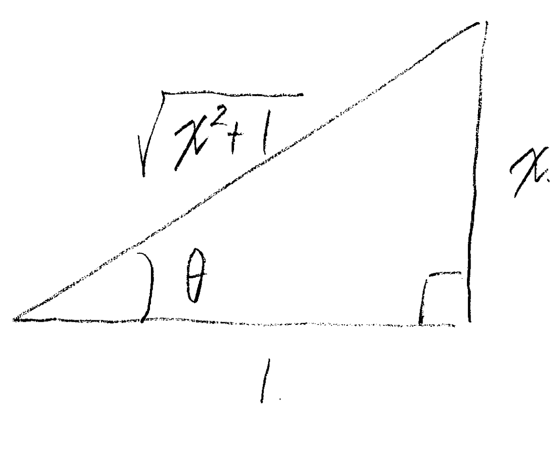
$$\left(\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta \, d\theta \\ \sqrt{x^2+1} &= \sec\theta \end{aligned} \right)$$

$$= \int \sec\theta \cdot \sec^2\theta \, d\theta$$

$$= \int \sec^3\theta \, d\theta$$

See 2.2 (iii)

$$= \frac{1}{2} \left\{ \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \right\} + C$$



$$= \frac{1}{2} \left\{ \sqrt{x^2+1} \cdot x + \ln \left| \sqrt{x^2+1} + x \right| \right\} + C$$

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(v) $$\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+1}}$$

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$$\left(\begin{array}{ll} x = \tan\theta & \theta \\ \sqrt{3} & \pi/3 \\ 1 & \pi/4 \\ dx = \sec^2\theta\, d\theta & \end{array}\right)$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\, d\theta}{\tan^2\theta \cdot \sec\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$\left(\begin{array}{lll} \theta & u = \sin\theta & du = \cos\theta\, d\theta \\ \pi/3 & \sqrt{3}/2 & \\ \pi/4 & \sqrt{2}/2 & \end{array}\right)$$

(41)

$$= \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{u^2}\, du$$

$$= \left[-\frac{1}{u} \right]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \left[\left(-\frac{2}{\sqrt{3}}\right) - \left(-\frac{2}{\sqrt{2}}\right) \right]$$

$$= \sqrt{2} - \frac{2}{3}\sqrt{3}$$

(vi) $$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$$u = x+1 \qquad u-1 = x$$
$$du = dx$$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$$\left(\begin{aligned} u &= 2\sin\theta \\ du &= 2\cos\theta\,d\theta \\ \sqrt{4-u^2} &= 2\cos\theta \end{aligned}\right)$$

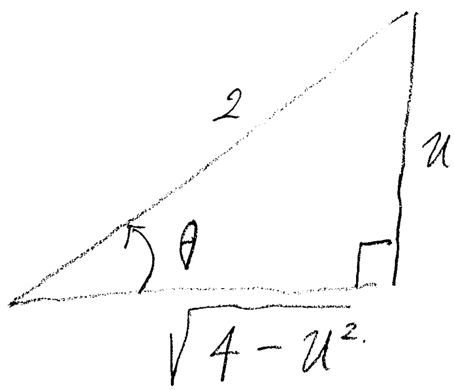
$$= \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\,d\theta$$

$$= \int (2 \sin\theta - 1)\, d\theta$$

$$= 2(-\cos\theta) - \theta + C$$

$$= -2\cos\theta - \theta + C$$

$$\frac{u}{2} = \sin\theta$$



$$= -\cancel{2} \cdot \frac{\sqrt{4-u^2}}{\cancel{2}} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

(vii) $\displaystyle\int \frac{x^2-2x+2}{\sqrt{x^2-2x+10}}\,dx$ 44

$$= \int \frac{(x-1)^2+1}{\sqrt{(x-1)^2+9}}\,dx$$

$$\begin{pmatrix} u = x-1 \\ du = dx \end{pmatrix}$$

$$= \int \frac{u^2+1}{\sqrt{u^2+9}}\,dx$$

$$\begin{pmatrix} u = 3\tan\theta \\ du = 3\sec^2\theta\,d\theta \\ \sqrt{u^2+9} = 3\sec\theta \end{pmatrix}$$

$$= \int \frac{\overset{9}{\cancel{3}}\tan^2\theta + 1}{\cancel{3\sec\theta}}\ \overset{\sec\theta}{\cancel{3\sec^2\theta}}\,d\theta$$

$$= \int \left(\underset{9}{\cancel{3}} \tan^2\theta + 1\right) \sec\theta \, d\theta$$

$$= \underset{9}{\cancel{3}} \int \tan^2\theta \sec\theta \, d\theta + \int \sec\theta$$

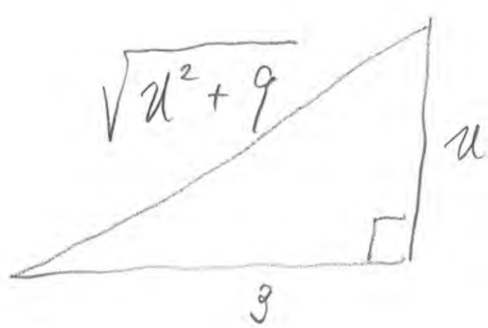
2.2 (iv)

$$= \underset{9}{\cancel{3}} \cdot \frac{1}{2} \left\{ \tan\theta \sec\theta - \ln|\sec\theta + \tan\theta| \right\}$$

$$+ \ln|\sec\theta + \tan\theta| + C$$

$$= \frac{\overset{9}{\cancel{3}}}{2} \tan\theta \sec\theta - \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$$

$$\frac{u}{3} = \tan\theta$$



46

$$= \frac{3}{2}\cdot\frac{u}{3}\cdot\frac{\sqrt{u^2+9}}{3} - \frac{1}{2}\ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + \underset{\substack{\shortparallel \\ C_{old}}}{C}$$

$$= \frac{1}{\cancel{6}\,2}\, u\sqrt{u^2+9} - \frac{1}{2}\ln\left|\sqrt{u^2+9} + u\right| + \underset{\substack{\shortparallel \\ C_{new}}}{C}$$

$$\left(C_{new} = C_{old} + \frac{1}{2}\ln 3 \right)$$

$$= \frac{1}{\cancel{6}\,2}(x-1)\sqrt{(x-1)^2+9}$$

$$- \frac{1}{2}\ln\left|\sqrt{(x-1)^2+9} + (x-1)\right| + C$$

(viii)

47

$$\int \frac{x}{\sqrt{3+2x-x^2}}\,dx$$

$$= \int \frac{x-1+1}{\sqrt{4-(x-1)^2}}\,dx$$

$$\left(\begin{array}{l} u = x-1 \\ du = dx \end{array} \right)$$

$$= \int \frac{u+1}{\sqrt{4-u^2}}\,du$$

$$\left(\begin{array}{l} u = 2\sin\theta \\ du = 2\cos\theta\,d\theta \\ \sqrt{4-u^2} = 2\cos\theta \end{array} \right)$$

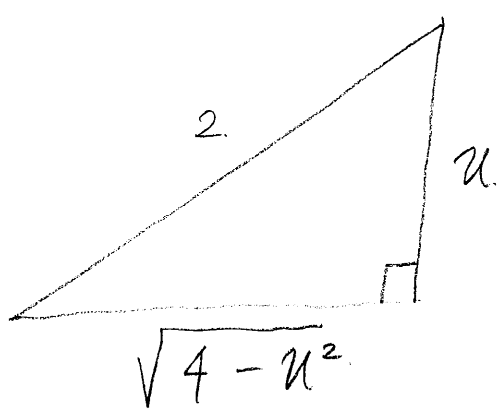
$$= \int \frac{2\sin\theta + 1}{\cancel{2\cos\theta}} \cancel{2\cos\theta}\, d\theta$$ 48

$$= \int (2\sin\theta + 1)\, d\theta$$

$$= 2(-\cos\theta) + \theta + C$$

$$= -2\cos\theta + \theta + C$$

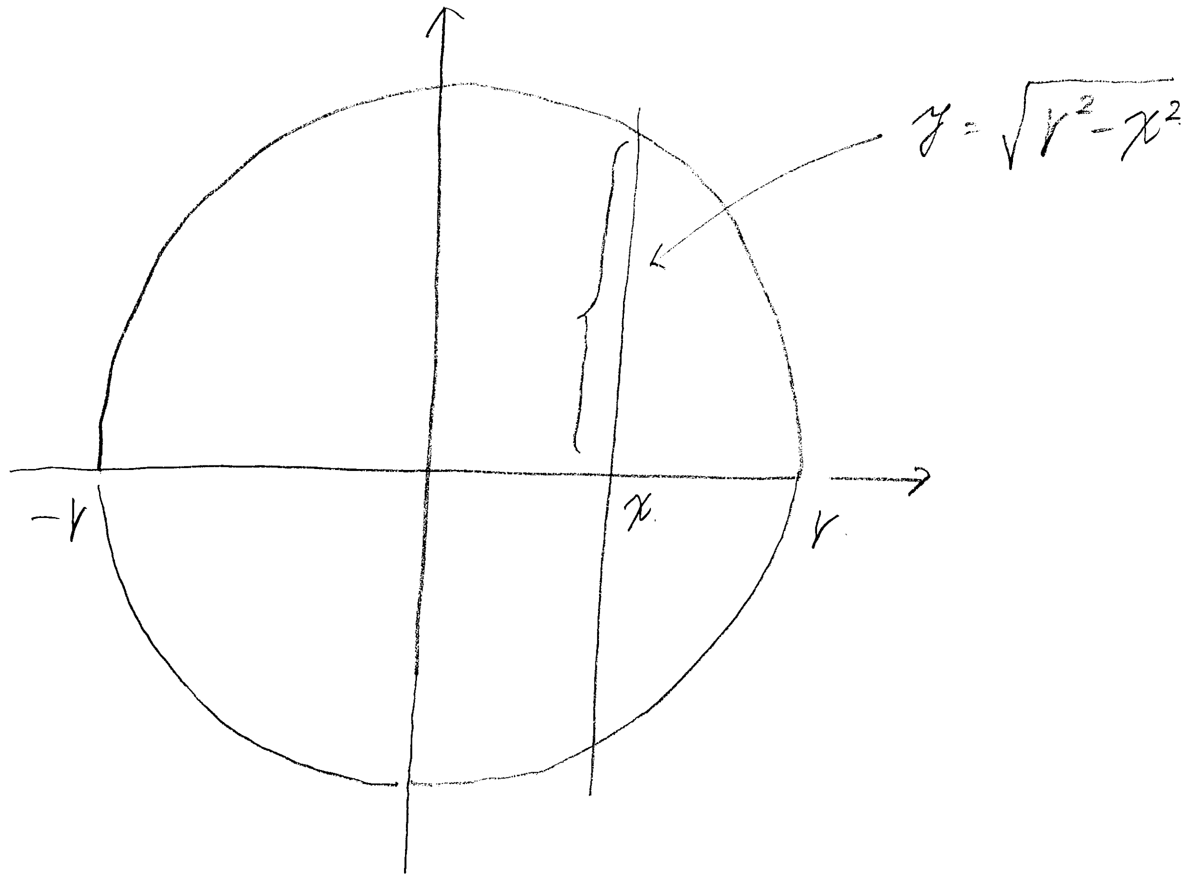
$$\frac{u}{2} = \sin\theta$$



$$= -\cancel{2} \cdot \frac{\sqrt{4-u^2}}{\cancel{2}} + \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x-1)^2} + \sin^{-1}\left(\frac{x-1}{2}\right) + C$$

3.2 added



$$A = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

θ	$x = r\sin\theta$	$\sin\theta = \frac{x}{r}$
$\pi/2$	r	1
$-\pi/2$	$-r$	-1

$$dx = r\cos\theta\, d\theta$$

added

$$\sqrt{r^2 - x^2} = r\cos\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} r\cos\theta \cdot r\cos\theta\, d\theta$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 2r^2 \cdot \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2}$$

$$= r^2\left[\pi/2 - \left(-\frac{\pi}{2}\right)\right] = \pi r^2$$

Note: $\frac{1}{2}\sin 2\cdot\frac{\pi}{2} = \frac{1}{2}\sin 2\cdot\left(-\frac{\pi}{2}\right) = 0$

4

~~3~~. 1.

49

(i)
$$\frac{1}{(x+2)(x^2-4)(x^2+x+1)^2}$$

$$\left(\begin{aligned} &x^2-4 = (x+2)(x-2) \\ &x^2+x+1 \\ &\quad = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \\ &\quad \text{indecomposable} \end{aligned}\right)$$

$$= \frac{1}{(x+2)^2(x-2)(x^2+x+1)^2}$$

$$= \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$+ \frac{C}{x-2}$$

$$+ \frac{Dx+E}{x^2+x+1} + \frac{Fx+G}{(x^2+x+1)^2}$$

(ii) $\dfrac{x^3}{(x-1)(x^3-1)(x^2+4x+5)}$ (50)

$$(x^3-1) = (x-1)\underbrace{(x^2+x+1)}_{\text{indecomposable}}$$

$$x^2+4x+5 = \underbrace{(x+2)^2+1}_{\text{indecomposable}}$$

$$= \frac{x^3}{(x-1)^2(x^2+x+1)(x^2+4x+5)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$+ \frac{Cx+D}{x^2+x+1}$$

$$+ \frac{Ex+F}{x^2+4x+5}$$

3.2. (51)

(i) $$\int \frac{x^2}{(x-1)^2}\,dx$$

I. Long division

$$\begin{array}{r|l} & \quad 1 \\ (x-1)^2 & \overline{)\, x^2} \\ \shortparallel & \quad x^2-2x+1 \\ x^2-2x+1 & \overline{\quad 2x-1} \end{array}$$

$$\frac{x^2}{(x-1)^2} = 1 + \frac{2x-1}{(x-1)^2}$$

II. Partial fractions

Step 1 factor $(x-1)^2$ ✓

Step 2 Type

$$\frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Step 3. Determine A & B

$$2x - 1 = A(x-1) + B$$

$\boxed{x = 1}$

$$1 = A \cdot 0 + B \rightarrow B = 1$$

$$2x - 1 = A(x-1) + 1$$

$$= Ax - A + 1$$

$$\rightarrow \begin{cases} 2 = A \\ -1 = -A + 1 \end{cases} \rightarrow A = 2$$

Note: Or you can compute

$$\frac{2x-1}{(x-1)^2} = \frac{2(x-1)+1}{(x-1)^2}$$

$$= \frac{\overset{= A}{2}}{x-1} + \frac{\overset{= B}{1}}{(x-1)^2}$$

$$\int \frac{2x-1}{(x-1)^2}\,dx$$ 53

$$= \int \left\{ \frac{2}{x-1} + \frac{1}{(x-1)^2} \right\} dx$$

$$= 2\ln|x-1| + \left(-\frac{1}{x-1}\right) + C$$

Final Answer :

$$\int \frac{x^2}{(x-1)^2}\,dx$$

$$= \int \left\{ 1 + \frac{2x-1}{(x-1)^2} \right\} dx$$

$$= x + 2\ln|x-1| - \frac{1}{x-1} + C$$

(ii) $$\int \frac{x+2}{x^2+2x+2}\,dx$$ 54

I. Long division ✓.

II. Partial fractions.

Step 1 factor

$$x^2+2x+2 = (x+1)^2+1$$

indecomposable

Step 2 Type

$$\frac{x+2}{x^2+2x+2} = \frac{Ax+B}{x^2+2x+2}$$

Step 3. Determine A & B.

Obviously $A = 1$ & $B = 2$

$$\int \frac{x+2}{x^2+2x+2}\,dx$$

55

$$= \int \frac{\frac{1}{2}(2x+2)+1}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$\int \frac{2x+2}{x^2+2x+2}\,dx$$

$$\left(\begin{array}{l} u = x^2+2x+2 \\ du = (2x+2)\,dx \end{array}\right)$$

$$= \int \frac{du}{u} = \ln|u| = \ln|x^2+2x+2| + C$$

$$= \ln(x^2+2x+2) + C$$

56

$$\int \frac{1}{x^2+2x+2}\,dx$$

$$= \int \frac{1}{(x+1)^2+1}\,dx$$

$$\left(\begin{array}{l} u = x+1 \\ du = dx \end{array} \right)$$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \tan^{-1}(u) + C$$

$$= \tan^{-1}(x+1) + C$$

Find answer

57

$$\int \frac{x+2}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\ln(x^2+2x+2) + \tan^{-1}(x+1) + C$$

(iii). $$\int \frac{x}{(x+1)(x-1)(x-2)}$$

58

I. Long division ✓

II. Partial fractions

Step 1 factor $(x+1)(x-1)(x-2)$ ✓

Step 2 Type

$$\frac{x}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

Step 3. Determine A, B, C

$$\begin{aligned} x = \; & A(x-1)(x-2) \\ & + B(x+1)(x-2) \\ & + C(x+1)(x-1) \end{aligned}$$

59

$\boxed{x = -1}$

$-1 = A(-2)(-3) \rightarrow A = -\frac{1}{6}$

$\boxed{x = 1}$

$1 = B \cdot 2(-1) \rightarrow B = -\frac{1}{2}$

$\boxed{x = 2}$

$2 = C \cdot 3 \cdot 1 \rightarrow C = \frac{2}{3}$

Final answer

$$\int \frac{x}{(x+1)(x-1)(x-2)}\, dx$$

$$= \int \left(\frac{-1/6}{x+1} + \frac{-1/2}{x-1} + \frac{2/3}{x-2} \right) dx$$

$$= -\frac{1}{6}\ln|x+1| - \frac{1}{2}\ln|x-1| + \frac{2}{3}\ln|x-2| + C$$

(iv) $$\int \frac{x^2}{(x-1)^2(x^2+1)}\,dx$$ 60

I. Long division ✓

II Partial fractions

Step 1 factor $(x-1)^2\underbrace{(x^2+1)}_{\text{indecomposable}}$ ✓

Step 2. Type

$$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$+ \frac{Cx+D}{x^2+1}$$

Step 3. Determine A, B, C, D.

$$x^2 = A(x-1)(x^2+1)$$
$$+ B(x^2+1)$$
$$+ (Cx+D)(x-1)^2$$

61

$\boxed{x = 1}$

$$1 = B \cdot 2 \qquad B = \frac{1}{2}$$

$$x^2 = A(x^3 - x^2 + x - 1)$$
$$+ \frac{1}{2}(x^2+1)$$
$$+ (Cx+D)(x^2-2x+1)$$

$$= (A+C)x^3 + \left(-A+\frac{1}{2}-2C+D\right)x^2$$
$$+ \left(-A+\frac{1}{2}+D\right)$$

$$\begin{cases} 0 = A + C \\ 1 = -A + \frac{1}{2} - 2C + D \\ 0 = -A + \frac{1}{2} + D \end{cases} \rightarrow \begin{cases} A = \frac{1}{2} \\ C = -\frac{1}{2} \\ D = 0 \end{cases}$$

$$\int \frac{x^2}{(x-1)^2(x^2+1)}\,dx$$

62

$$= \int \left\{ \frac{1/2}{x-1} + \frac{1/2}{(x-1)^2} + \frac{-1/2\,x}{x^2+1} \right\} dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\left(-\frac{1}{x-1}\right)$$

$$- \frac{1}{2}\cdot\frac{1}{2}\ln(x^2+1) + C$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2(x-1)}$$

$$- \frac{1}{4}\ln(x^2+1) + C$$

(v) $$\int \frac{x^2+x+2}{x^2+4x+5}\,dx$$

~~66~~ 63

I. Long division

$$\begin{array}{r} 1 \\ x^2+4x+5 \enclose{longdiv}{x^2+x+2} \\ x^2+4x+5 \\ \hline -3x-3 \end{array}$$

$$\int \frac{x^2+x+2}{x^2+4x+5}\,dx$$

$$= \int \left\{ 1 + \frac{-3x-3}{x^2+4x+5} \right\} dx$$

$$= \int 1\,dx - 3\int \frac{x+1}{x^2+4x+5}\,dx$$

Concentrate on computing

$$\int \frac{x+1}{x^2+4x+5}\,dx$$

II. Partial fractions

Step 1 factor $x^2 + 4x + 5$

$$x^2 + 4x + 5 = (x+2)^2 + 1$$

indecomposable

Step 2 Type

$$\frac{x+1}{x^2+4x+5} = \frac{\overset{=1}{A}x + \overset{=1}{B}}{x^2+4x+5}$$

Step 3. Determine A & B

Obviously $A = 1$, $B = 1$

$$\int \frac{x+1}{x^2+4x+5}\,dx = \int \frac{\frac{1}{2}(2x+4) - 1}{x^2+4x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{1}{x^2+4x+5}\,dx$$

$\cancel{64}$

65

$$\int \frac{2x+4}{x^2+4x+5}\,dx$$

$$\left(\begin{array}{l} u = x^2+4x+5 \\ du = (2x+4)\,dx \end{array}\right)$$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|x^2+4x+5| + C$$

$$= \ln(x^2+4x+5) + C$$

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$$

$$\left(\begin{array}{l} u = x+2 \\ du = dx \end{array}\right)$$

$$= \int \frac{1}{u^2+1}\,du = \tan^{-1}(u) + C$$

$$= \tan^{-1}(x+2) + C$$

Final answer

~~65~~

66

$$\int \frac{x^2 + x + 2}{x^2 + 4x + 5}\, dx$$

$$= \int 1\, dx - 3 \int \frac{x+1}{x^2+4x+5}\, dx$$

$$= \int 1 \cdot dx - 3 \left\{ \frac{1}{2} \int \frac{2x+4}{x^2+4x+5}\, dx - \int \frac{1}{x^2+4x+5}\, dx \right\}$$

$$= x - \frac{3}{2} \ln \left(x^2 + 4x + 5\right) + 3 \tan^{-1}(x+2) + C$$

5 ~~4~~. 1.

67

(i) $$\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx$$

$$= \lim_{b\to\infty} \int_0^{b} \frac{e^x}{e^{2x}+1}\,dx$$

$$\left(\begin{array}{lll} x & u = e^x & du = e^x dx \\ b & e^b & \\ 0 & 1 & \end{array}\right)$$

$$= \lim_{b\to\infty} \int_1^{e^b} \frac{1}{u^2+1}\,du$$

$$= \lim_{b\to\infty} \left[\tan^{-1} u\right]_1^{e^b}$$

$$= \lim_{b\to\infty} \left[\underbrace{\tan^{-1}(\underset{\downarrow \atop \infty}{e^b})}_{\downarrow \atop \pi/2} - \underset{\pi/4}{\underset{\shortparallel}{\tan^{-1}(1)}}\right] = \frac{\pi}{4}$$

(iii) $\displaystyle\int_0^\infty \frac{e^{2x}}{e^{2x}+1}\,dx$ (68)

$$= \lim_{b\to\infty}\int_0^b \frac{e^{2x}}{e^{2x}+1}\,dx$$

$$\left(\begin{array}{lll} x & u = e^{2x}+1 & du = 2e^{2x}dx \\ b & e^{2b}+1 & \\ 0 & 2 & \end{array}\right)$$

$$= \lim_{b\to\infty}\int_2^{e^{2b}+1} \frac{\frac{1}{2}\,du}{u}$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\underbrace{\ln(\underbrace{e^{2b}+1}_{\downarrow\ \infty})}_{\downarrow\ \infty} - \ln 2\Big]$$

$$= \infty$$

(iii) $$\int_0^9 \frac{1}{x-1}\,dx$$

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^9 \frac{1}{x-1}\,dx$$

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{x-1}\,dx \qquad = \lim_{c\to 1^+} \int_c^9 \frac{1}{x-1}\,dx$$

$$= \lim_{b\to 1^-} \Big[\ln|x-1|\Big]_0^b \qquad = \lim_{c\to 1^+} \Big[\ln|x-1|\Big]_c^9$$

$$= \underbrace{\lim_{b\to 1^-} \Big[\underbrace{\ln \underbrace{|b-1|}_{\downarrow\, 0^+}}_{-\infty} - \ln 1\Big]}_{=\,-\infty} \qquad = \underbrace{\lim_{c\to 1^+} \Big[\ln 8 - \underbrace{\ln \underbrace{|c-1|}_{\downarrow\, 0^+}}_{-\infty}\Big]}_{=\,+\infty}$$

Final conclusion

$$\int_0^9 \frac{1}{x-1}\,dx \quad \text{diverges.}$$

Warning: Do NOT conclude

$$\int_0^9 \frac{1}{x-1}\,dx = -\infty + \infty = 0$$

or

$$\int_0^9 \frac{1}{x-1}\,dx = \Big[\ln|x-1|\Big]_0^9$$

$$= \ln 8 - \ln 1$$

$$= \ln 8$$

(iv) $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

$$= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{b\to 1^-} \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^b = \lim_{b\to 1^-} \frac{3}{2}\left[\underbrace{(b-1)^{\frac{2}{3}}}_{\to 0} - 1\right] = -\frac{3}{2}$$

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{c\to 1^+} \int_c^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{c\to 1^+} \int \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_c^9 = \lim_{c\to 1^+} \frac{3}{2}\left[4 - \underbrace{(c-1)^{\frac{2}{3}}}_{\to 0}\right] = 6$$

$$= -\frac{3}{2} + 6 = \frac{9}{2}$$

72

(v) $$\int_{-\infty}^{\infty} x\,dx$$

$$= \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$$

$$\lim_{b \to -\infty} \int_{b}^{0} x\,dx \qquad \lim_{c \to +\infty} \int_{0}^{c} x\,dx$$

$$\lim_{b \to -\infty} \left[\frac{x^2}{2}\right]_{b}^{0} \qquad \lim_{c \to +\infty} \left[\frac{x^2}{2}\right]_{0}^{c}$$

$$\lim_{b \to -\infty} \left[\frac{0^2}{2} - \frac{b^2}{2}\right] \qquad \lim_{c \to +\infty} \left[\frac{c^2}{2} - \frac{0^2}{2}\right]$$

$$-\infty \qquad +\infty$$

Final conclusion

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$$\int_{-\infty}^{\infty} x\,dx \quad \text{diverges.}$$

Warning:

① Do NOT conclude

$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$$

$$= -\infty + \infty = 0.$$

② Do NOT conclude

$$\int_{-\infty}^{\infty} x\,dx = \lim_{b\to\infty}\int_{-b}^{b} x\,dx = \lim_{b\to\infty}\left[\frac{x^2}{2}\right]_{-b}^{b}$$

$$= \lim_{b\to\infty}\left[\frac{b^2}{2} - \frac{(-b)^2}{2}\right]$$

$$= 0$$

(vi) $$\int_0^\infty x e^{-x}\, dx$$ 74

$$= \lim_{b \to \infty} \int_0^b x e^{-x}$$

$$\int x e^{-x} dx = \int u\, dv$$

$$\left(\begin{array}{ll} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x} dx \end{array} \right)$$

$$= uv - \int v\, du$$

$$= x(-e^{-x}) - \int (-e^{-x})\, dx$$

$$= -xe^{-x} - e^{-x} + C$$

$$= -(x+1)e^{-x} = -\frac{x+1}{e^x}$$

$$= \lim_{b \to \infty} \left[-\frac{x+1}{e^x} \right]_0^b$$

$$= \lim_{b \to \infty} \left[\underbrace{\left(-\frac{b+1}{e^b} \right)}_{\downarrow \atop 0} - \left(-\frac{0+1}{e^0} \right) \right]$$

$$= 1.$$

Note:

$$\lim_{b \to \infty} \frac{b+1}{e^b} \overset{\text{L.H.}}{=} \lim_{b \to \infty} \frac{1}{e^b} = 0$$

(vi) $$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

$$= \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$$

$$= \lim_{b \to -\infty} \int_{b}^{0} x e^{-x^2} dx \qquad = \lim_{c \to \infty} \int_{0}^{c} x e^{-x^2} dx$$

$$\left(\begin{array}{cc} x & u = -x^2 \\ 0 & 0 \\ b & -b^2 \\ \multicolumn{2}{c}{du = -2x\,dx} \end{array} \right) \qquad \left(\begin{array}{cc} x & u = -x^2 \\ c & -c^2 \\ 0 & 0 \\ \multicolumn{2}{c}{du = -2x\,dx} \end{array} \right)$$

$$= \lim_{b \to -\infty} \int_{-b^2}^{0} -\frac{1}{2} e^{u} du \qquad = \lim_{c \to \infty} \int_{0}^{c} -\frac{1}{2} e^{u} du$$

$$= \lim_{b \to -\infty} -\frac{1}{2} \left[e^{u} \right]_{-b^2}^{0} \qquad = \lim_{c \to \infty} -\frac{1}{2} \left[e^{u} \right]_{0}^{-c^2}$$

$$= \qquad\qquad\qquad\qquad\qquad =$$

$$\underbrace{\lim_{b \to -\infty} -\frac{1}{2}\left[e^{0} - \underbrace{e^{-b^2}}_{\to 0}\right]}_{= \; -\frac{1}{2}} \qquad \underbrace{\lim_{c \to \infty} -\frac{1}{2}\left[\underbrace{e^{-c^2}}_{\to 0} - e^{0}\right]}_{= \; +\frac{1}{2}}$$

Final conclusión

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

Warning:

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

$$= \lim_{b \to \infty} \underbrace{\int_{-b}^{b} x e^{-x^2} dx}_{\substack{= \\ 0}}$$

0 since xe^{-x^2} odd.

$$= 0$$

is a FAKE argument!

i.e. invalid

6 ~~5~~.1

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(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{(-1)^n n}{n^2+1} = 0$

since

$$-\frac{n}{n^2+1} \le \frac{(-1)^n n}{n^2+1} \le \frac{n}{n^2+1}$$

$n\to\infty$: $\quad -\dfrac{n}{n^2+1} \to 0, \quad \dfrac{(-1)^n n}{n^2+1} \to 0, \quad \dfrac{n}{n^2+1} \to 0$

by Squeeze Th.

(ii)

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \tan^{-1}\Big(\underbrace{\frac{n^3+5}{n^2}}_{\downarrow\ \infty}\Big) = \pi/2$$

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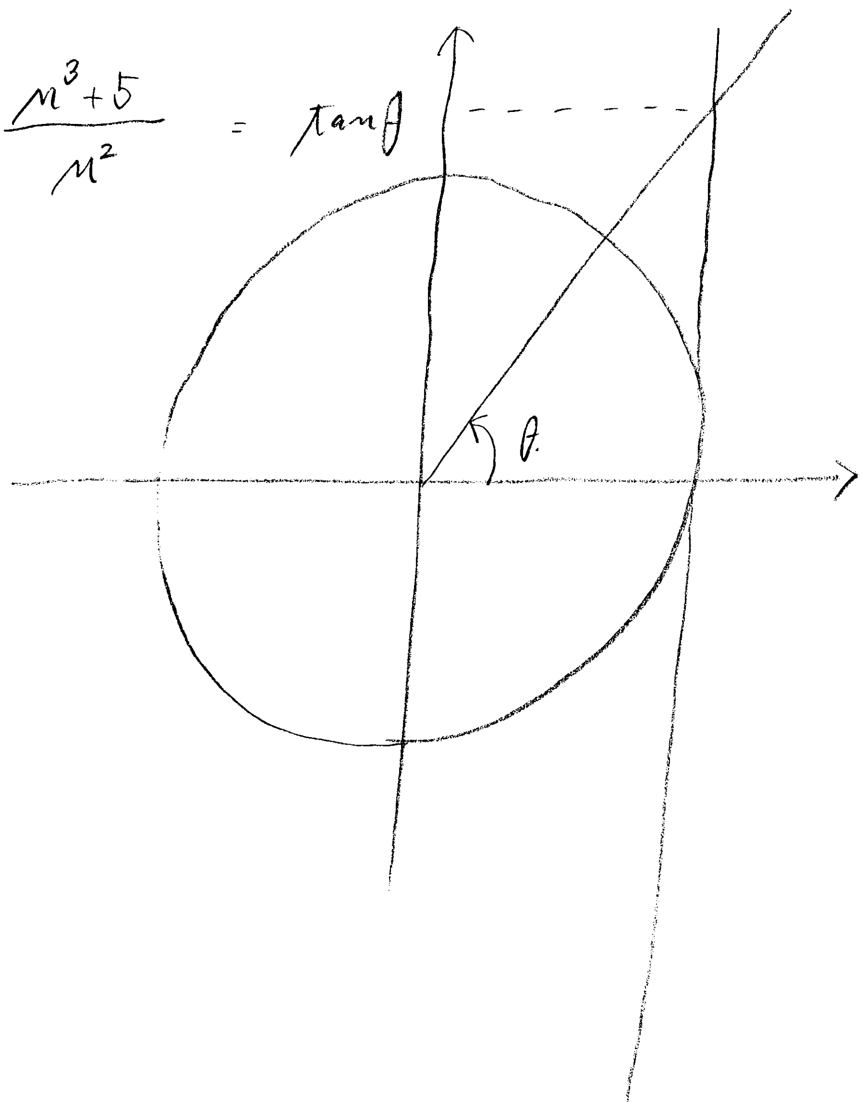
$$\frac{n^3+5}{n^2} = \tan\theta$$



(iii) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \underbrace{\sin\underbrace{\left(\frac{\pi}{2}-\frac{1}{n}\right)}_{\to\ \pi/2}}_{\to\ 1}$$

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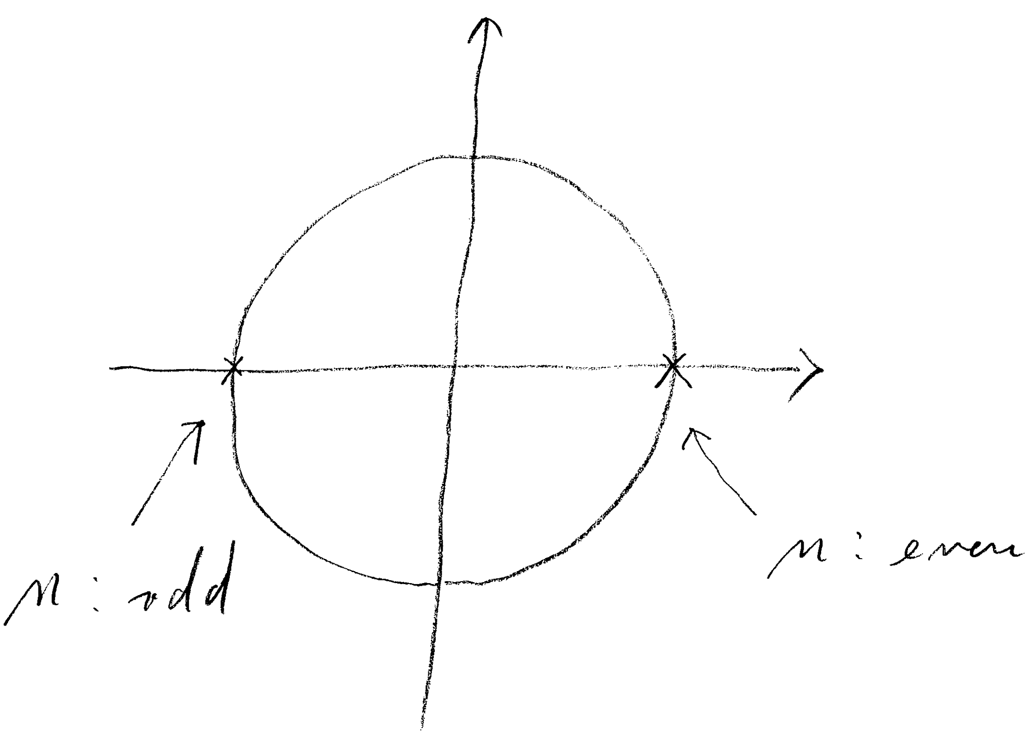
DNE

(iv) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi) \qquad \text{DNE}$$



(v) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sin(n\pi) = 0$

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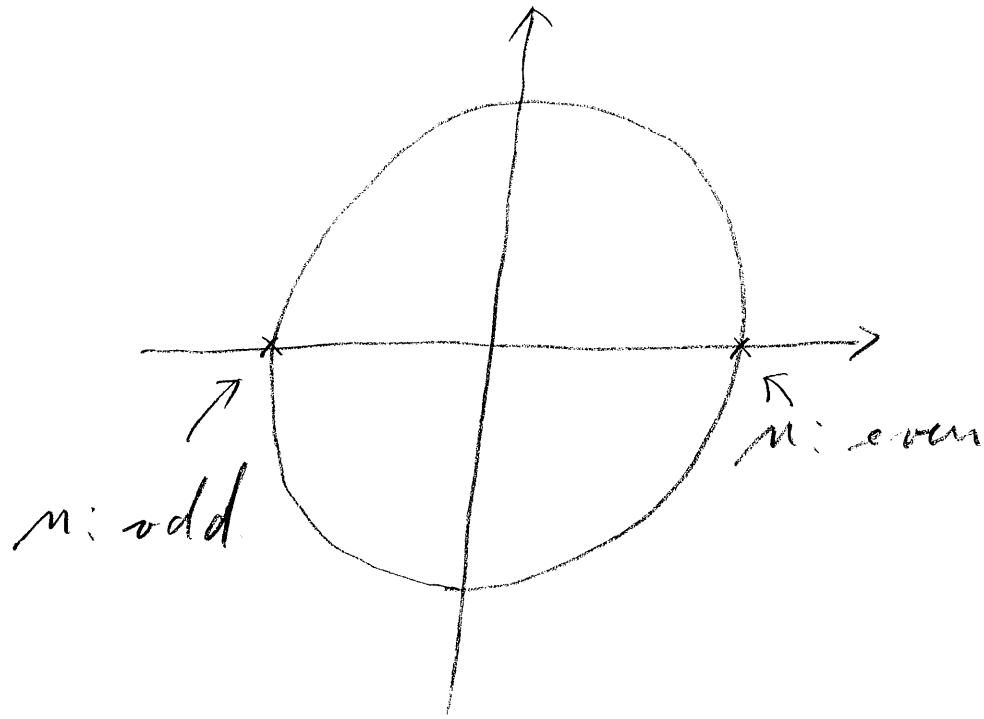

(vi) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n!\, n^2}{(n+2)!} \cos\left(\dfrac{1}{2n}\right)$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\; n^2}{\cancel{n!}\,(n+1)(n+2)} \cos\left(\frac{1}{2n}\right)$$

$$= \lim_{n\to\infty} \underbrace{\frac{n^2}{(n+1)(n+2)}}_{\downarrow \atop 1} \underbrace{\cos\Big(\underbrace{\frac{1}{2n}}_{\downarrow \atop 0}\Big)}_{\downarrow \atop 1} = 1$$

(vii)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = 1.$$

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since

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = 1.$$