

Strategy for Trigonometric Integration

TYPES

1. $\int \sin^m x \cos^n x dx$
2. $\int \tan^m x \sec^n x dx$
3. $\left\{ \begin{array}{l} \int \sin mx \cos nx dx \\ \int \sin mx \sin nx dx \\ \int \cos mx \cos nx dx \end{array} \right.$

TYPE 1. Case: m odd or n odd

Attitude: Smile ! (Easy !)

Strategy (Say, n is odd. The case where m is odd is identical. If both m and n are odd, you can go either way.): Extract one $\cos x$ to see $\cos x dx$ at the end. Then use the substitution $u = \sin x$ via the equality $\cos^2 x = 1 - \sin^2 x$.

Example:

$$\begin{aligned} \int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cdot \cos x dx \\ &= \int \sin^4 x (\cos^2 x)^2 \cdot \cos x dx \\ &= \int \sin^4 x (1 - \sin^2 x)^2 \cdot \cos x dx \\ &\quad \{(\text{Substitution}) u = \sin x \ \& \ du = \cos x dx\} \\ &= \int u^4 (1 - u^2)^2 \cdot du \\ &= \int (u^4 - 2u^6 + u^8) du \\ &= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \\ &= \frac{\sin^5 x}{5} - \frac{2 \sin^7 x}{7} + \frac{\sin^9 x}{9} + C \end{aligned}$$

Case: m even and n even

Attitude: Not smile but not frown, either. (Not so good but not so bad, either.)

Strategy: Using the double angle formula

$$\begin{aligned}\sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos^2 x &= \frac{1 + \cos 2x}{2}\end{aligned}$$

drop the degree.

Example:

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx \\ &= \frac{1}{4} \int (1 - \cos^2 2x) dx \\ &= \frac{1}{4} \int \left(1 - \frac{1 + \cos 4x}{2} \right) dx \\ &= \frac{1}{8} \int (1 - \cos 4x) dx \\ &= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C \\ &= \frac{1}{8} x - \frac{1}{32} \sin 4x + C\end{aligned}$$

TYPE 2. **Case: n even, $n \neq 0$**

Attitude: Smile ! (Easy !)

Strategy: Extract $\sec^2 x$ to see $\sec^2 x dx$ at the end. Then use the substitution $u = \tan x$ via the equality $\sec^2 x = 1 + \tan^2 x$ to turn the remaining even powers of $\sec x$ into the expression in terms of $\tan x$.

Example:

$$\begin{aligned}
 \int \tan^3 x \sec^4 x dx &= \int \tan^3 x \sec^2 x \cdot \sec^2 x dx \\
 &= \int \tan^3 x (1 + \tan^2 x) \cdot \sec^2 x dx \\
 &\quad \{(\text{Substitution}) u = \tan x \ \& \ du = \sec^2 x dx\} \\
 &= \int u^3 (1 + u^2) du \\
 &= \int (u^3 + u^5) du \\
 &= \frac{u^4}{4} + \frac{u^6}{6} + C
 \end{aligned}$$

Case: $n = 0$

Attitude: Not smile but not frown, either. (Not so good but not so bad, either.)

$$\boxed{m = 1}$$

$$\begin{aligned}
 \int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
 &\quad \{(\text{Substitution}) u = \cos x \ \& \ du = -\sin x dx\} \\
 &= \int \frac{1}{u} (-du) \\
 &= - \int \frac{1}{u} du \\
 &= -\ln |u| + C \\
 &= -\ln |\cos x| + C \\
 &= \ln |\cos x|^{-1} + C \\
 &= \ln |\sec x| + C
 \end{aligned}$$

$$\boxed{m > 1}$$

Strategy: Drop the number m using the equality $\tan^2 x = \sec^2 x - 1$.

Examples:

$$\begin{aligned} m = 3 : \int \tan^3 x dx &= \int \tan x \tan^2 x dx \\ &= \int \tan x (\sec^2 x - 1) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \end{aligned}$$

We already know how to compute $\int \tan x dx$.

We can compute $\int \tan x \sec^2 x dx$ by using integration by substitution $u = \tan x$ and $du = \sec^2 x dx$

$$\begin{aligned} \int \tan x \sec^2 x dx &= \int u du \\ &= \frac{u^2}{2} + C \\ &= \frac{\tan^2 x}{2} + C \end{aligned}$$

$$\begin{aligned} m = 5 : \int \tan^5 x dx &= \int \tan^3 x \tan^2 x dx \\ &= \int \tan^3 x (\sec^2 x - 1) dx \\ &= \int \tan^3 x \sec^2 x dx - \int \tan^3 x dx \end{aligned}$$

We already know how to compute $\int \tan^3 x dx$.

We can compute $\int \tan^3 x \sec^2 x dx$ by using integration by substitution $u = \tan x$ and $du = \sec^2 x dx$

$$\begin{aligned}
 \int \tan^3 x \sec^2 x dx &= \int u^3 du \\
 &= \frac{u^4}{4} + C \\
 &= \frac{\tan^4 x}{4} + C
 \end{aligned}$$

Case: m odd & n odd

Attitude: Smile ! (Easy !)

Strategy: Extract $\tan^x \sec x$ to see $\tan x \sec x dx$ at the end. Then use the substitution $u = \sec x$ via the equality $\tan^2 x = \sec^2 x - 1$ to turn the remaining even powers of $\tan x$ into the expression in terms of $\sec x$.

Example:

$$\begin{aligned}
 \int \tan^3 x \sec^5 x dx &= \int \tan^2 x \sec^4 x \cdot \tan x \sec x dx \\
 &= \int (\sec^2 x - 1) \sec^4 x \cdot \tan x \sec x dx \\
 &\quad \{(\text{Substitution}) u = \sec x \ \& \ du = \tan x \sec x dx\} \\
 &= \int (u^2 - 1)u^4 du \\
 &= \int (u^6 - u^4) du \\
 &= \frac{u^7}{7} - \frac{u^5}{5} + C
 \end{aligned}$$

Case: m even & n odd

Attitude: Hard ! (Frowning !)

Note: By using the equality $\tan^2 x = \sec^2 x - 1$, one can express $\tan^m x$ part only in terms of (even powers of) $\sec x$.

Therefore, the problem is reduced to computing $\int \sec^n x dx$ with n being odd (the case of n being even is already discussed).

$$\boxed{n = 1}$$

$$\begin{aligned} \int \sec x dx &= \int \frac{\sec x (\tan x + \sec x)}{(\sec x + \tan x)} dx \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

$$\boxed{n > 1 \text{ \& } n \text{ odd}}$$

Strategy: Using integration by parts, drop the number n .

Example:

$$\begin{aligned} \int \sec^3 x dx &= \int u dv \\ \text{(Substitution)} \quad &\begin{cases} u = \sec x & \& v = \tan x \\ du = \sec x \tan x dx & \& dv = \sec^2 x dx \end{cases} \\ &= uv - \int v du \\ &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx - \int \sec x dx \end{aligned}$$

Therefore, moving $-\int \sec^3 x dx$ on the right-hand side to the left-hand side, we conclude

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx.$$

That is to say,

$$\int \sec^3 x dx = \frac{1}{2} \left(\sec x \tan x + \int \sec x dx \right),$$

where we already know how to compute $\int \sec x dx$.

$$\begin{aligned} \int \sec^5 x dx &= \int u dv \\ \text{(Substitution)} \quad &\begin{cases} u = \sec^3 x & \& v = \tan x \\ du = 3 \sec^3 x \tan x dx & \& dv = \sec^2 x dx \end{cases} \\ &= uv - \int v du \\ &= \sec^3 x \tan x - \int 3 \sec^3 x \tan^2 x dx \\ &= \sec^3 x \tan x - 3 \int \sec^3 x (\sec^2 x - 1) dx \\ &= \sec^3 x \tan x - 3 \int \sec^5 x dx + 3 \int \sec^3 x dx \end{aligned}$$

Therefore, moving $-3 \int \sec^5 x dx$ on the right-hand side to the left-hand side, we conclude

$$4 \int \sec^5 x dx = \sec^3 x \tan x + 3 \int \sec^3 x dx.$$

That is to say,

$$\int \sec^5 x dx = \frac{1}{4} \left(\sec^3 x \tan x + 3 \int \sec^3 x dx \right),$$

where we already know how to compute $\int \sec^3 x dx$.

TYPE 3. Attitude: Smile ! (Easy !)

Strategy: Using the formulas

$$\begin{cases} \sin mx \cos nx &= \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \\ \sin mx \sin nx &= \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \\ \cos mx \cos nx &= \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] \end{cases}$$

we split the multiplication of sin and cos into the addition.

Example:

$$\begin{aligned} \int \sin 4x \cos 5x dx &= \int \frac{1}{2} [\sin(4-5)x + \sin(4+5)x] dx \\ &= \frac{1}{2} \int [\sin(-x) + \sin 9x] dx \\ &= \frac{1}{2} \int [-\sin x + \sin 9x] dx \\ &= \frac{1}{2} \left[\cos x - \frac{1}{9} \cos 9x \right] + C \\ &= \frac{1}{2} \cos x - \frac{1}{18} \cos 9x + C \end{aligned}$$