Detailed Plan of Lectures for MA 166

$\boxed{\text{Lesson 1}}$ the beginning of the semester

Topics: Vectors in the plane & Vectors in the 3-dimensional space
Section Number: 13.1, 13.2
Lecture Plan:

(1) Use the first 20 mimutes to explain the ground rules.
- Structure of the course
- MyLabMath Homework
- Recitation Class (Pre-Quiz Exercise Problems)

(2) Vectors in general
- Question: What is a vector ?
  Answer: An arrow: direction and magnitude
  (followed by the explanation of tail, head), and the zero vector $\vec{0}$
- Two vectors are "equal" as long as you can move one to the other by parallel transform

(3) Operations on (among) the vectors
- Scalar multiplication
- Addition
- Subtraction
- Basic properties of the operations (as stated on Page 811 on the textbook)

(4) Vectors in the plane
- expression by the components
- formula for the magnitude
- formulas associated with the operations
- Discussion of the unit vector

(5) Vectors in the 3-dimensional space
- expression by the components
- formula for the magnitude
- formulas associated with the operations

$\boxed{\text{Lesson 2}}$

Topics: Geometry in the plane and in the 3-dimensional space
Section Number: 13.1, 13.2, 13.5
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 1. This should serve as a review for Lesson 1.
   - Review for Lesson 1
   - MyLabMath Homework for Lesson 1

(2) Equations of the planes parallel to the coordinate axes
(3) Equation of a circle (brief discussion)
(4) Equation of a sphere
   - Distance formula
   - Equation of a sphere with center (a, b, c) and radius r
     - ◦ $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
     - ◦ How to find the center and radius, given the equation

   Example: $x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$

(5) Equation of a line
   - in 2-dimensional space
     - ◦ parametric one using the parameter t
     - ◦ the one in terms of x, y

   Example: passing $P = (1, 3)$ with directional vector $\vec{v} = \langle 5, 2\rangle$
   - in 3-dimensional space
     - ◦ parametric one using the parameter t
     - ◦ the one in terms of x, y

   Example: passing $P = (1, 3, 4)$ with directional vector $\vec{v} = \langle 5, 2, 3\rangle$

Lesson 3

Topics: Dot Product
Section Number: 13.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 2. This should serve as a review for Lesson 2.
- Review for Lesson 2
- MyLabMath Homework for Lesson 2

(2) Definition of the dot product
- 2-dimensional case
  - $\vec{u} = \langle u_1, u_2 \rangle, \vec{v} = \langle v_1, v_2 \rangle, \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

  Example: $\vec{u} = \langle 4, 1 \rangle$, $\vec{v} = \langle 3, 4 \rangle$, $\vec{u} \cdot \vec{v} = 4 \cdot 3 + 1 \cdot 4 = 16$
- 3-dimensional case
  - $\vec{u} = \langle u_1, u_2, u_3 \rangle, \vec{v} = \langle v_1, v_2, v_3 \rangle, \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

  Example: $\vec{u} = \langle 4, 1, -2 \rangle$, $\vec{v} = \langle 3, 4, 5 \rangle$, $\vec{u} \cdot \vec{v} = 4 \cdot 3 + 1 \cdot 4 + (-2) \cdot 5 = 6$

(3) Geometric meaning of the dot product
- Theorem: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ - Option: Proof using Law of Cosines

(4) Applications

Orthogonality condition
- $\vec{u} \perp \vec{v} \Longleftrightarrow \vec{u} \cdot \vec{v} = 0$
- Application: Equation of a plane

Example Problems

1 Find the equation of a plane passing $(1, 3, 2)$ and orthogonal to $\vec{v} = \langle 2, 1, 5 \rangle$

2 Given the equation of the plane $2x - 3y - z = 6$,

(i) Find a point on the plane.

(ii) Find a vector orthogonal to the plane.

Projection of a vecor $\vec{u}$ onto $\vec{v}$
- Explanation using a picture
  - direction: $\vec{v}$
  - magnitude: $|\vec{u}| \cos\theta$
  - the unit vector in the direction of $\vec{v}$: $\dfrac{\vec{v}}{|\vec{v}|}$

  $\longrightarrow$ Formula: $\text{proj}_{\vec{v}}\vec{u} = |\vec{u}| \cos\theta \dfrac{\vec{v}}{|\vec{v}|}$
- Formula using the dot product

$$\left\{ \begin{array}{lcl} \text{proj}_{\vec{v}}\vec{u} & = & |\vec{u}| \cos\theta \dfrac{\vec{v}}{|\vec{v}|} \\ & = & \dfrac{|\vec{u}||\vec{v}| \cos\theta}{|\vec{v}||\vec{v}|} \vec{v} \\ & = & \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \end{array} \right.$$

Note: $\text{scal}_{\vec{v}}\vec{u} = |\text{proj}_{\vec{v}}\vec{u}| = |\vec{u}| \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \dfrac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{v} \cdot \vec{v}}}$

Lesson 4

Topics: Cross Product
Section Number: 13.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 3. This should serve as a review for Lesson 3.
- Review for Lesson 3
- MyLabMath Homework for Lesson 3

(2) Definition of the determinant
- 2×2 case

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad = bc$$

- 3×3 case

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\
&= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}$$

(3) Definition of the cross product $\vec{u} \times \vec{v}$
Example: $\vec{u} = \langle -1, 0, 6 \rangle, \vec{v} = \langle 2, -5, -3 \rangle$

$$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} 0 & 6 \\ -5 & -6 \end{vmatrix}, - \begin{vmatrix} -1 & 6 \\ 2 & -3 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix} \right\rangle$$

- Observation $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

(4) Geometric meaning of the cross product $\vec{u} \times \vec{v}$
- direction
  - $\vec{u}, \vec{v} \perp \vec{u} \times \vec{v}$
  - Right Hand Rule
- magnitude $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

Example:

$$\begin{cases} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{cases}$$

(5) Another expression for the cross product $\vec{u} \times \vec{v}$
- Example: $\vec{u} = \langle -1, 0, 6 \rangle, \vec{v} = \langle 2, -5, -3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix}$$

Lesson 5

Topics: Applications of the cross product and computation of the area of the region between cyrves

Section Number: 13.4, 6.2

Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 4. This should serve as a review for Lesson 4.
- Review for Lesson 4
- MyLabMath Homework for Lesson 4

(2) Applications of the cross product
- Equation of a plane

Example: Find the equation of the plane
passing $P = (5, -2, 3)$ and
containing $\vec{u} = \langle -1, 0, 6\rangle$ and $\vec{v} = \langle 2, -5, -3\rangle$
- Condition for being parallel $\vec{u} \,//\, \vec{v} \Longleftrightarrow \vec{u} \times \vec{v} = \vec{0}$

(3) Computing the area of the region between two curves
- Explanation using a picture
- Formula

$$\int_a^b L(x)\ dx = \int_a^b \{f(x) - g(x)\}\ dx \quad \text{assuming} \quad f(x) \geq g(x) \quad \text{over} \quad [a, b]$$
$$\int_a^b L(x)\ dx = \int_a^b |f(x) - g(x)|\ dx \quad \text{in general}$$

- Examples

Example Problem 1: Find the area of the region enclosed by $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$.

Example Problem 2 (Optional): Find the area of the region enclosed by $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$.

Example Problem 3 (MUST): Find the area of the region in the 1st quadrant enclosed by $y = x^{2/3}$ and $y = x - 4$.

Draw the pictures, and explain

Method 1: Integration with respect to x.

Method 2: Integration with respect to y. (EASIER !)

Lesson 6

Topics: Volumes by slicing, Washer method
Section Number: 6.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 5. This should serve as a review for Lesson 5.
- Review for Lesson 5
- MyLabMath Homework for Lesson 5

(2) Computing the volume by slicing
- Explain the general idea using a picture
- Formula

$$V = \int_a^b A(x)\ dx = \sum A(x)\Delta x \quad \text{(as the Riemann sum)}$$

where
  - ∘ $A(x)$ is the area of the cross section, and where
  - ∘ $A(x)\ dx$ (or $A(x)\ \Delta x$) represents the volume of the thin slice.
- Review: Computing the area by slicing

$$A = \int_a^b L(x)\ dx = \sum L(x)\Delta x \quad \text{(as the Riemann sum)}$$

where
  - ∘ $L(x)$ is the length of the cross section, and where
  - ∘ $L(x)\ dx$ (or $L(x)\ \Delta x$) represents the area of the thin slice.
- Example Problem: Find the volume of the solid
  - ∘ whose base is the region enclosed by $y = 1 - x^2$ in the 1st quadrant,
  - ∘ whose cross section parallel to the y-axis and perpendicular to the base

is the square.

(3) Washer method
- Explain the idea using the example problems

Example Problem 1: Find the volume of the solid obtained by rotating the region R about the x-axis

Description of R: the region bounded by

$$\begin{cases} y &= & f(x) = (x+1)^2, \\ x &= & 0, \\ x &= & 2. \end{cases}$$

Example Problem 2: Find the volume of the solid obtained by rotating the region R about the x-axis

Description of R:
the region bounded by

$$\begin{cases} y &= & f(x) = \sqrt{x}, \\ y &= & f(x) = x^2 \end{cases}$$

Lesson 7

Topics: Volumes by shells
Section Number: 6.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 6. This should serve as a review for Lesson 6.
  - Review for Lesson 6
  - MyLabMath Homework for Lesson 6

(2) Computing the volume by cylindrical shells
  - Explain the idea by going over the example problems
  - Formula

$$\int_a^b 2\pi x f(x)\,dx \quad \text{in case rotated around } y-\text{axis} \quad \text{(See Example Problem 1)}$$
$$\int_a^b 2\pi y f(y)\,dy \quad \text{in case rotated around } x-\text{axis} \quad \text{(See Example Problem 2)}$$

Question: What should we do in case the solid is obtained by rotating the region around some different axis ? (See Example Problem 3)
  - Example Problems

Example Problem 1: Find the volume of the solid obtained by rotating the region R about the y-axis
Description of R: the region in the 1st quadrant bounded by

$$\begin{cases} y &= f(x) = \sin(x^2), \\ y &= 0 \ \text{ i.e., x} - \text{axis}, \\ x &= \sqrt{\dfrac{\pi}{2}}. \end{cases}$$

Example Problem 2: Find the volume of the solid obtained by rotating the region R about the x-axis
Description of R: the region in the 1st quadrant bounded by

$$\begin{cases} y &= f(x) = 2x - x^2, \\ y &= x. \end{cases}$$

Example Problem 3: Find the volume of the solid obtained by rotating the region R **about the line** $x = -\dfrac{1}{2}$.
Description of R: the region in the 1st quadrant bounded by

$$\begin{cases} y &= f(x) = \sqrt{x}, \\ y &= 1. \end{cases}$$

(3) FAQ: How can we tell wwhich method to use ?

ANSWER: One can use BOTH Washer and Shell methods (in principle). Sometmes it is more difficult to compute the volume using one method than the other.

  - Show how to compute the volume of the solid in Example Problem 1, leading to the computation of

$$\int \sin^{-1} y\,dy.$$

Lesson 8

Topics: Lengths of curves and Surface area
Section Number: 6.5, 6.6
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 7. This should serve as a review for Lesson 7.
- Review for Lesson 7
- MyLabMath Homework for Lesson 7

(2) How to compute the length of a curve $y = f(x) \quad a \leq x \leq b$
- Explain the general idea using a picture
- Formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \{f'(x)\}^2}\, dx$$

- Example Problems

Example Problem 1: Find the (arc) length of the curve

$$y = f(x) = x^{3/2} \quad 0 \leq x \leq 4.$$

Example Problem 2 (Super Difficult ! Optional !): Find the (arc) length of the curve

$$y = f(x) = \ln(x + \sqrt{x^2 - 1}) \quad 1 \leq x \leq \sqrt{2}.$$

Method 1 (Integration with respect to x): $L = \int_1^{\sqrt{2}} \sqrt{1 + \{f'(x)\}^2}\, dx$

Troubles:

① Integration is improper !
② Integration is hard to compute !

Method 2 (Integration with respect to y EASIER !):

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + \{g'(y)\}^2}\, dy$$

Note:

$$\left\{\begin{array}{lcl} y & = & f(x) = \ln(x + \sqrt{x^2 - 1}) \quad 1 \leq x \leq \sqrt{2} \\ \longrightarrow & & \\ x & = & g(y) = \dfrac{e^y + e^{-y}}{2} \quad 0 \leq y \leq \ln(\sqrt{2} + 1) \end{array}\right.$$

Formula:

$$L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \int_c^d \sqrt{\{g'(y)\}^2 + 1}\, dy$$

Show the trick to reduce $\sqrt{1 + \{g'(y)\}^2}$ to a non-square-root form.

(3) How to compute the area of a surface obtained by revolving the curve

$$y = f(x) \quad a \le x \le b$$

around the x-axis.

- How to compute the area of the frustrum
- Explain the general idea using a picture
- Formula

$$A = \int_a^b 2\pi f(x)\sqrt{1 + \{f'(x)\}^2}\ dx.$$

- Example Problem: Check that the surface area of a sphere of radius r is given by $A = 4\pi r^2$, realizing that the sphere is obtained by revolving the curve

$$y = f(x) = \sqrt{r^2 - x^2} \quad -r \le x \le r$$

around the x-axis.

Lesson 9

Topics: Physical applications
Section Number: 6.7
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 8. This should serve as a review for Lesson 8.
- Review for Lesson 8
- MyLabMath Homework for Lesson 8

(2) Compute the work to strech the spring
- Hooke's Law: $F(x) = kx$
  - Discussion of how to determine k
- Formula

$$W = \int_a^b F(x)\ dx = \int_a^b kx\ dx$$

• Example Problem: When you need to exert 10 N of force to stretch the spring $x = 0.1$ m from the natural length, compute the work to be done to strech the spring from $x = 0.1$ m to $x = 0.35$ m.

(3) Compute the work to lift the chain
- Explan the idea using a picture
- Example Problem: A 10 m-long chain with density of 1.5 kg/m hangs from a platform at a construction site taht is 11 m above the ground.

(1) Find the work required to lift the chain to the platform.
(2) Find the work to bring the bottom end of the chain to the platform so that the chain is folded into half.

(4) Compute the work to pump out the water
- Explan the idea using a picture
- Example Problems

Example Problem 1: Compute the work required to pump out all the water in the cylindrical tank of radius $r = 5$ m and height 15 m from the top. Use the number $\rho = 1000$ kg/m^3 for the density of the water, and $g = 9.8$ m/s^2 for the gravitational acceleration.

Example Problem 2: A cylindrical tank with a length of 10 m and a radius of 5 m is on its side and half full of gassoline. How much work is reqruired to empty the tank through an outlet pipe at the top of the tank ? Use the number $\rho = 737$ kg/m^3 for the density of the gasoline, and $g = 9.8$ m/s^2 for the gravitational acceleration.

(5) Compute the force on a dam (Optional)
- Explan the idea using a picture
- Example Problem: A large vertical dam in the shape of a symmetric trapezoid has a height of 30 m, a width of 20 m at its base, and a width of 40 m at the top. What is the total force on the face of the dam when the reservoir is full ?

Lesson 10

Topics: Integration by parts
Section Number: 8.2
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 9. This should serve as a review for Lesson 8.
- Review for Lesson 9
- MyLabMath Homework for Lesson 9

(2) Formula

$$\int u dv = uv - \int v du$$

- Derive the formula from the product rule

$$(uv)' = u'v + uv'$$

(3) Example Problems

Example Problem 1: $\int xe^x \, dx$

$$\begin{cases} u &= x &, & v &= e^x \\ du &= dx &, & dv &= e^x dx \end{cases}$$

Example Problem 2: $\int x \sin x \, dx$

$$\begin{cases} u &= x &, & v &= -\cos x \\ du &= dx &, & dv &= \sin x dx \end{cases}$$

Example Problem 3: $\int \ln x \, dx$

$$\begin{cases} u &= \ln x &, & v &= x \\ du &= \dfrac{1}{x} dx &, & dv &= dx \end{cases}$$

Example Problem 4: $\int \sin^{-1} x \, dx$

$$\begin{cases} u &= \sin^{-1} x &, & v &= x \\ du &= \dfrac{1}{\sqrt{1-x^2}} dx &, & dv &= dx \end{cases}$$

(4) **FAQ: How should we choose** u **&** dv **?**

Give us an algorithm to choose the proper u & dv !

Theoretical ANSWER:

① There is NO such algorithm.
② All the choices are equally valid as formulas.
(Integration by parts is a parallel transform !)
③ One choice may lead to a “more difficult” problem.

Example: $\int xe^x \, dx$

Choose

$$\begin{cases} u &= e^x &, & v &= \dfrac{1}{2} x^2 \\ du &= e^x dx &, & dv &= x dx \end{cases}$$

Then

$$\begin{aligned}\int xe^x\ dx &= \int udv\\ &= uv-\int vdu\\ &= e^x\cdot\frac{1}{2}x^2-\int\frac{1}{2}x^2e^xdx\\ &= e^x\cdot\frac{1}{2}x^2-\frac{1}{2}\int x^2e^xdx\end{aligned},$$

while

$\int x^2e^xdx$ is "more difficult" than $\int xe^xdx$.

Practical ANSWER:

① Choose dv so that you can compute the antiderivative v easily.

② Make sure $\int vdu$ is simpler than the priginal $\int udv$.

One more FAQ: What should I do if I make a wrong choice ?

ANSWER: Don't worry ! Go back, and work with another choice ($^\vee\circ^\vee$)

(5) Interesting problems: No matter what choice you make, the problem dose not seem to become any simpler.

- Example Problem: $\int e^x\cos x\ dx$

Step 1. Apply Integration by Parts to $\int e^x\cos xdx$.

$$\begin{cases} u = e^x &,\quad v = \sin x\\ du = e^xdx &,\quad dv = \cos xdx\end{cases}$$

$$\begin{aligned}\int e^x\cos xdx &= \int udv\\ &= uv-\int vdu\\ &= e^x\sin x-\int\sin xe^xdx\\ &= \sin xe^x-\int e^x\sin xdx\end{aligned}$$

Step 2. Apply Integration by Parts one more time to $\int e^x\sin xdx$.

$$\begin{cases} u = e^x &,\quad v = -\cos x\\ du = e^xdx &,\quad dv = \sin xdx\end{cases}$$

$$\begin{aligned}\int e^x\sin xdx &= \int udv\\ &= uv-\int vdu\\ &= e^x(-\cos x)-\int e^x(-\cos x)dx\\ &= -e^x\cos x+\int e^x\cos xdx\end{aligned}$$

- Step 3. Combine Steps 1 and 2.

$$\begin{aligned}\int e^x \cos x dx &= e^x \sin x - \int e^x \sin x dx \\ &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x dx \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx.\end{aligned}$$

Oh, NO. We came back to the same integration we started with !
Happy Conclusion !
Moving $-\displaystyle\int e^x \cos x dx$ on the right hand side to the left, we get

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C = e^x(\sin x + \cos x) + C$$

and hence

$$\boxed{\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C.}$$

Note: We replaced the old "C" with the new "C", which is $\frac{1}{2}$ of the old "C".

$\boxed{\text{Lesson 11}}$

Topics: Trigonometric Integration Part 1
Computaion of the integration of the form

$$\int \sin^m x \cos^n x \, dx$$

Section Number: 8.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 10. This should serve as a review for Lesson 10.
- Review for Lesson 10
- MyLabMath Homework for Lesson 10

(2) **Case: m odd or n odd (Smiley face case ($^\vee \circ ^\vee$))**
- **Strategy**:

① Take one from the odd power one and combine it with dx, say, $\sin x dx$ (resp. $\cos x dx$).

② Then apply the substitution $u = \cos x$ (resp. $u = \sin x$), using the relation $\sin^2 x = 1 - \cos^2 x$ (resp. $\cos^2 x = 1 - \sin^2 x$).
- Example Problems

Example Problem 1: $\int \sin^3 x \cos^2 x \, dx$

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \, \sin x \, dx \\
& \quad (u = \cos x, du = -\sin x \, dx) \\
&= \int (1 - \cos^2 x) \cos^2 x \, \sin x \, dx \\
&= \int (1 - u^2) u^2 \, (-du) \\
&= \int (u^4 - u^2) \, du \\
&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\
&= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C
\end{aligned}$$

Example Problem 2: $\int \cos^3 x \, dx = \int \sin^0 x \cos^3 x \, dx$

$$\begin{aligned}
\int cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\
& \quad (u = \sin x, du = \cos x \, dx) \\
&= \int (1 - \sin^2 x) \cos x \, dx \\
&= \int (1 - u^2) du \\
&= u - \frac{1}{3} u^3 + C \\
&= \sin x - \frac{1}{3} \sin^3 x + C
\end{aligned}$$

(3) **Case: Otherwise (i.e., m even and n even (Frowning face case (T$\vee$T))**.

- **Strategy**: Use half angle formula to reduce the total degree
- Example Problem

$\int \sin^2 x \cos^2 x \, dx$ (total degree $= 4$)

Use

$$\begin{cases} \sin^2 x &= \dfrac{1-\cos 2x}{2} \\ \cos^2 x &= \dfrac{1+\cos 2x}{2} \end{cases}$$

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\ &= \frac{1}{4}\int (1-\cos^2 2x) \, dx \end{aligned}$$

Since the last one has total degree $= 2(< 4)$, it is "simpler" than before !

In fact, we know how to compute it as follows:

$$\begin{cases} \displaystyle\int 1 \, dx &= x + C \\ \displaystyle\int \cos^2 2x \, dx &= \displaystyle\int \frac{1+\cos 4x}{2} \, dx \\ &= \dfrac{1}{2}x + \dfrac{1}{8}\sin 4x + C \end{cases}$$

Finally we have

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \frac{1}{4}\int (1-\cos^2 2x) \, dx \\ &= \frac{1}{4}\left[\int 1 \, dx - \int \cos^2 2x \, dx\right] \\ &= \frac{1}{4}\left[x - \left(\frac{1}{2}x + \frac{1}{8}\sin 4x\right)\right] + C \\ &= \frac{1}{8}x - \frac{1}{32}\sin 4x + C \end{aligned}$$

Lesson 12

Topics: Trigonometric Integration Part 2
Computaion of the integration of the form

$$\int \tan^m x \sec^n x \; dx$$

Section Number: 8.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 10. This should serve as a review for Lesson 11.
- Review for Lesson 11
- MyLabMath Homework for Lesson 11

(2) **Case:** m **whatever and** $n > 0$ **even**
- **Strategy**:

① Take 2 from even $n > 0$ and combine it with dx to have $\sec^2 x dx$.
② Then apply the substitution $u = \tan x$ ($du = \sec^2 x dx$), using the relation $\sec^2 x = 1 + \tan^2 x$.

- Example Problem: $\displaystyle\int \tan^3 x \sec^4 x \; dx$

$$\begin{aligned}
\int \tan^3 x \sec^4 x \; dx &= \int \tan^3 x \sec^2 x \;\; \sec^2 x \; dx \\
&\quad (u = \tan x, du = \sec^2 x dx) \\
&= \int \tan^3 x (1 + \tan^2 x) \;\; \sec^2 x \; dx \\
&= \int u^3 (1 + u^2) \; du \\
&= \int (u^3 + u^5) \; du \\
&= \frac{1}{4} u^4 + \frac{1}{6} u^6 + C \\
&= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C
\end{aligned}$$

(3) **Case:** m **odd and** $n > 0$ **whatever**
- **Strategy**:

① Take $\tan x \sec x$ and combine it with dx to have $\tan x \sec x dx$.
② Then apply the substitution $u = \sec x$ ($du = \tan x \sec x dx$), using the relation $\tan^2 x = \sec^2 x - 1$.

- Example Problem: $\displaystyle\int \tan^3 x \sec^3 x \; dx$

$$\begin{aligned}
\int \tan^3 x \sec^3 x \; dx &= \int \tan^2 x \sec^2 x \;\; \tan x \sec x \; dx \\
&\quad (u = \sec x, du = \tan x \sec x \; dx) \\
&= \int (sec^2 x - 1) \sec^2 x \;\; \tan x \sec x dx \\
&= \int (u^2 - 1) u^2 \; du \\
&= \int (u^4 - u^2) \; du \\
&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\
&= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C
\end{aligned}$$

(4) **Case:** $n = 0$

- **Strategy**:

① When $m = 1$, we compute $\tan x \; dx = -\ln|\cos x| + C = \ln|\sec x| + C$.

② When $m > 1$, use $\tan^2 = \sec^2 x - 1$. For the fisrt part, use substitution $u = \tan x$. For the second part, observe m has decreased and hence it has become simpler.

- Example Problem: $\int \tan^3 x \; dx$

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \;\; \tan^2 x \; dx \\
&= \int \tan x (\sec^2 x - 1) \; dx \\
&= \int \tan x \sec^2 x \; dx - \int \tan x \; dx \\
& \quad (u = \tan x \text{ for the first part}), (m = 1 < 3 \text{ for the second part}) \\
&= u \; du - \int \tan x \; dx \\
&= \frac{1}{2} u^2 - \textstyle\int \tan x \; dx \\
&= \frac{1}{2} \tan^2 - \ln|\sec x| + C
\end{aligned}$$

(5) **Case: Otherwise i.e.,** m **even and** $n > 0$ **odd**

- **Strategy**:

① Using $\tan^2 x = \sec^2 x - 1$, reduce the problem to the case where $m = 0$.

② When $m > 1$, use Integration by Parts to decrease m so that the problem becomes simpler.

③ When $m = 1$, we have

$$\sec x \; dx = \ln|\sec x + \tan x| + C.$$

- Example Problem: $\int \tan^2 x \sec x dx$

①

$$\begin{aligned}
\int \tan^2 x \sec x \; dx &= \int (\sec^2 x - 1) \sec x dx \\
&= \int \sec^3 x \; dx - \int \sec x \; dx
\end{aligned}$$

② We only have to compute $\int \sec^3 x \; dx$, since we know $\int \sec x \; dx$ from ③.

$$\int \sec^3 x \; dx = \int \sec x \sec^2 x \; dx$$

$$\left\{ \begin{array}{rclcrcl} u &=& \sec x &,& v &=& \tan x \\ du &=& \tan x \sec x \; dx &,& dv &=& \sec^2 x dx \end{array} \right.$$

$$\begin{aligned}
\int \sec^3 x \; dx &= \textstyle\int \sec x \sec^2 x \; dx = \displaystyle\int u dv \\
&= uv - vdu \\
&= \sec x \tan x - \int \tan x \tan x \sec x \; dx \\
&= \sec x \tan x - \int \tan^2 x \sec x \; dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \; dx \\
&= \sec x \tan x - \int \sec x \; dx - \int \sec^3 x \; dx
\end{aligned}$$

Moving $-\int \sec^3 x \; dx$ on the right hand side to the left, we obtain

$$2\int \sec^3 x \; dx = \sec x \tan x - \int \sec x \; dx$$

and hence

$$\int \sec^3 x \; dx = \frac{1}{2}\left(\sec x \tan x - \int \sec x \; dx\right)$$

③ Finally we have

$$\begin{aligned}
\int \tan^2 x \sec x \; dx &= \int \sec^3 x \; dx - \int \sec x \; dx \\
&= \frac{1}{2}\left(\sec x \tan x - \int \sec x \; dx\right) - \textstyle\int \sec x \; dx \\
&= \frac{1}{2}\left(\sec x \tan x - 3\int \sec x \; dx\right) \\
&= \frac{1}{2}\left(\sec x \tan x - 3\ln|\sec x + \tan x|\right) + C.
\end{aligned}$$

Memo:

Distribute the document "Strategy for Trigonometric Integration".

Lesson 13

Topics: Trigonometric Substitutions Part 1
Section Number: 8.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 12. This should serve as a review for Lesson 12.
- Review for Lesson 12
- MyLabMath Homework for Lesson 12

(2) Explain the Main Theme: Kill "$\sqrt{}$" in the integration.
Note: There are cases where we use our trigonometric substitution for othe than killing "$\sqrt{}$".

(3) Summmary of 3 Types of the Trigonometric Substitution
- Type I: $\sqrt{a^2 - x^2}$ ($a > 0$: some constant)

$$\begin{cases} x &= a\sin\theta \\ dx &= a\cos\theta d\theta \\ \sqrt{a^2 - x^2} &= a\cos\theta \end{cases}$$

- Type II: $\sqrt{a^2 + x^2}$ ($a > 0$: some constant)

$$\begin{cases} x &= a\tan\theta \\ dx &= a\sec^2\theta d\theta \\ \sqrt{a^2 + x^2} &= a\sec\theta \end{cases}$$

- Type III: $\sqrt{x^2 - a^2}$ ($a > 0$: some constant)

$$\begin{cases} x &= a\sec\theta \\ dx &= a\sec\theta\tan\theta d\theta \\ \sqrt{x^2 - a^2} &= a\tan\theta \end{cases}$$

(4) Discussion of Type I by Example Problem
- Type I: $\displaystyle\int \frac{dx}{(16 - x^2)^{3/2}} = \int \frac{dx}{(16 - x^2)\sqrt{16 - x^2}}$

$$\begin{cases} x &= 4\sin\theta \\ dx &= 4\cos\theta d\theta \\ \sqrt{4^2 - x^2} &= 4\cos\theta \end{cases}$$

$$\begin{aligned} \int \frac{dx}{(16 - x^2)\sqrt{16 - x^2}} &= \int \frac{4\cos\theta d\theta}{\{16 - (4\sin\theta)^2\}4\cos\theta} \\ &= \int \frac{d\theta}{16 - 16\sin^2\theta} = \int \frac{d\theta}{16(1 - \sin^2\theta)} \\ &= \frac{1}{16}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{16}\int \sec^2\theta d\theta \\ &= \frac{1}{16}\tan\theta + C \end{aligned}$$

How to go back from the variable "θ" to the variable "x"

$\circ$ Draw the picture of the right triangle with
hypotenuse $= 4$ and vertical $= x \longrightarrow$ bottom $= \sqrt{16 - x^2}$
Final Conclusion:

$$\int \frac{dx}{(16 - x^2)^{3/2}} = \frac{1}{16}\tan\theta + C = \frac{1}{16}\frac{x}{\sqrt{16 - x^2}} + C.$$

Lesson 14

Topics: Trigonometric Substitutions Part 2
Section Number: 8.4
Lecture Plan:

(1) Use the first 20 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 13. This should serve as a review for Lesson 13.
- Review for Lesson 13
- MyLabMath Homework for Lesson 13

(2) Discussion of Type II and Type III by Example Problems
- Type II: $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

$$\left\{\begin{array}{ccc} x & = & 3\tan\theta \\ dx & = & 3\sec 2\theta d\theta \\ \sqrt{9+x^2} & = & 3\sec\theta \end{array}\right.$$

$$\begin{aligned} \int \frac{dx}{\sqrt{9+x^2}} &= \int \frac{3\sec^2\theta d\theta}{3\sec\theta} = \int \sec\theta d\theta \\ &= \ln|\sec\theta+\tan\theta|+C \end{aligned}$$

How to go back from the variable "θ" to the variable "x"

∘ Draw the picture of the right triangle with
base $=3$ and vertical $=x \longrightarrow$ hypotenuse $=\sqrt{9+x^2}$
Final Conclusion:

$$\begin{aligned} \int \frac{dx}{\sqrt{9+x^2}} &= \ln|\sec\theta+\tan\theta|+C = \ln|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}|+C \\ &= \ln\frac{1}{3}|\sqrt{9+x^2}+x|+C = \ln\frac{1}{3}+\ln|\sqrt{9+x^2}+x|+C \\ &= \ln|\sqrt{9+x^2}+x|+C. \end{aligned}$$

Note: In the last step, we set the "C_{new}" to be equal to $\ln\frac{1}{3}+C_{\text{old}}$".

- Type III: $\displaystyle\int \frac{\sqrt{x^2-25}}{x}dx$

$$\left\{\begin{array}{ccc} x & = & 5\sec\theta \\ dx & = & 5\sec x\tan x\theta d\theta \\ \sqrt{x^2-25} & = & 5\tan\theta \end{array}\right.$$

$$\begin{aligned} \int \frac{\sqrt{x^2-25}}{x}dx &= \int \frac{5\tan\theta}{5\sec\theta}5\sec\theta\tan\theta d\theta \\ &= \int 5\tan^2\theta d\theta = 5\int(\sec^2\theta-1)d\theta \\ &= 5(\tan\theta-\theta)+C. \end{aligned}$$

How to go back from the variable "θ" to the variable "x"

∘ Draw the picture of the right triangle with
hypotenuse $=x$ and base $=5 \longrightarrow$ vertical $=\sqrt{x^2-25}$
Final Conclusion:

$$\begin{aligned} \int \frac{\sqrt{x^2-25}}{x}dx &= 5(\tan\theta-\theta)+C \\ &= 5\left(\frac{\sqrt{x^2-25}}{5}-\sec^{-1}\left(\frac{x}{5}\right)\right)+C \end{aligned}$$

Challenging Questions:

① Is it true $\sec^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{\sqrt{x^2-25}}{x}\right)$?

② Can I replace $\sec^{-1}\left(\frac{x}{5}\right)$ above with $\sin^{-1}\left(\frac{\sqrt{x^2-25}}{x}\right)$?

Lesson 15

Topics: Summary of Trigonometric Integrations and Trigonometric Substitutions
Section Number: 8.1, 8.2, 8.3, 8.4
Lecture Plan:

The subjects of Trigonometric Integrations and Trigonometric Substitutions are formidable both in quantity and difficulty for the students to digest. Most likely the instructor cannot cover everything scheduled to be covered in Lessons 11, 12, 13, 14. This lesson is reserved as a shock absorber so that the instructor can catch up with the schedule and/or review the materials.

$\boxed{\text{Lesson 16}}$

Topics: Partial Fractions Part 1: How to compute the integration of the form

$$\int \frac{Q(x)}{P(x)}\,dx$$

where $Q(x)$ and $P(x)$ are polynomials

Section Number: 8.5

Lecture Plan:

(1) Explain the algorithm to compute the integration of the form $\displaystyle\int \frac{Q(x)}{P(x)}\,dx$ using the following example problem

• Example Problem: $\displaystyle\int \frac{x^4 - 2x^3 + 2x^2 + 9x - 2}{x^3 - x^2 - 2x}\,dx$

Part I: Long division

$$x^4 - 2x^3 + 2x^2 + 9x - 2 = (x-1)(x^3 - x^2 - 2x) + 3x^2 + 7x - 2$$

and hence

$$\int \frac{x^4 - 2x^3 + 2x^2 + 9x - 2}{x^3 - x^2 - 2x}\,dx = \int (x-1)\,dx + \int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}\,dx$$

◦ $\displaystyle\int (x-1)\,dx$ Easy ! $\longrightarrow$ Concentrate on computing $\displaystyle\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}\,dx$

Part II: Partial Fractions Compute $\displaystyle\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}\,dx$

Step 1: Factor the denominator $x^3 - x^2 - 2x$

$$x^3 - x^2 - 2x = x(x+1)(x-2)$$

Step 2: Determine the type

$$\frac{3x^2 + 7x - 2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

Step 3: Determine the coefficients A, B, C

Multiply $x(x+1)(x-2)$ to the above equatuion to get

$$\begin{array}{rcll} 3x^2 + 7x - 2 & = & & A(x+1)(x-2) \\ & & + & Bx(x-2) \\ & & + & Cx(x+1) \\ & = & & (A+B+C)x^2 \\ & & + & (-A-2B+C)x \\ & & + & (-2A+0+0)1 \end{array}$$

Method 1: Solve the equations

$$\begin{cases} A + B + C & = & 3 \\ -A - 2B + C & = & 7 \\ -2A & = & -2. \end{cases}$$

Method 2 (Easier !):

Plug in the values $x = 0, -1, 2$

$\boxed{x = 0}$ $\quad -2 = A \cdot (-2) \longrightarrow A = 1$

$\boxed{x = -1}$ $\quad -6 = B \cdot 3 \longrightarrow B = -2$

$\boxed{x = 2}$ $\quad -24 = C \cdot 6 \longrightarrow C = 4$

Step 4

$$\begin{aligned}
\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx &= \int \frac{3x^2+7x-2}{x(x+1)(x-2)} \\
&= \int \left\{ \frac{1}{x} + \frac{-2}{x+1} + \frac{4}{x+2} \right\} dx \\
&= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + C.
\end{aligned}$$

Part III: Final Conclusion

$$\begin{aligned}
\int \frac{x^4-2x^3+2x^2+9x-2}{x^3-x^2-2x}\,dx &= \int (x-1)\,dx + \int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx \\
&= \frac{1}{2}x^2 - x + \ln|x| - 2\ln|x+1| + 4\ln|x-2| + C.
\end{aligned}$$

(2) Discuss the Exercise Problem (Optional): $\displaystyle\int \frac{9x^2+2x-1}{(x-1)(2x^2+7x-4)}\,dx$

Lesson 17

Topics: Partial Fractions Part 2: How to compute the integration of the form

$$\int \frac{Q(x)}{P(x)}\,dx$$

where $Q(x)$ and $P(x)$ are polynomials

Section Number: 8.5

Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 16. This should serve as a review for Lesson 16.
- Review for Lesson 16
- MyLabMath Homework for Lesson 16

(2) Explain the types of Partial Fractions by example
- Introduce "Indecomposable Quadratic Forms"
  - How to judge a given quadratic form is decomposable or indecomposable by completing the square
- Example

$$\begin{aligned}
\frac{\bigcirc}{(x-1)(x+2)(x^2+5x+6)(x^2+5)^2(x^2+2x+3)} &= \frac{\bigcirc}{(x-1)(x+2)^2(x+3)(x^2+5)^2\{(x+1)^2+2\}} \\
&= \frac{A}{x-1} \\
&+ \frac{B}{x+2} + \frac{C}{(x+2)^2} \\
&+ \frac{D}{x+3} \\
&+ \frac{Ex+F}{x^2+5} + \frac{Gx+H}{x^2+5} \\
&+ \frac{Ix+J}{x^2 2x+3}
\end{aligned}$$

(3) Discuss the example problem reviewing the alogorithm in Lesson 16
- Example Problem: $\displaystyle\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$
  - How to compute the integration associated to an indecomposable quadratic factor

Part I: Long division Already done ✓

Part II: Partial Fractions

Step 1: Factor the denominator $(x-2)(x^2-2x+3)$. Already done ✓

Note: (x^2-2x+3) is an indecomposable quadratic form.

Step 2: Determine the type

$$\frac{7x^2-13x+13}{(x-2)(x^2-2x+3} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}$$

Step 3: Determine the coefficients A, B, C

Multiply $(x-2)(x^2-2x+3)$ to the above equatuion to get

$$\begin{aligned}
7x^2-13x+13 &= A(x^2-2x+3) \\
&+ (Bx+C)(x-2)
\end{aligned}$$

Method 2:

Plug in the values $x = 2$ and any other two distinct values, say, $x = 0, 1$

$\boxed{x = 2}$ $\quad -15 = A \cdot 3 \longrightarrow A = 5$

$\boxed{x = 0}$ $\quad 13 = 15 + C \cdot (-2) \longrightarrow C = 1$

$\boxed{x = 1}$ $\quad 7 = 10 + (B + 1) \cdot (01) \longrightarrow B = 2$

Step 4

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \int \left(\frac{5}{x-2} + \frac{2x+1}{x^2 - 2x + 3} \right) dx$$

- $\circ$ $\displaystyle\int \frac{5}{x-2}\, dx = 5 \ln|x - 2| + C$ Easy to compute !
- $\circ$ How should we compute $\displaystyle\int \frac{2x+1}{x^2 - 2x + 3}\, dx$? (Difficult !)

$$\begin{aligned}
\int \frac{2x+1}{x^2 - 2x + 3}\, dx &= \int \frac{(2x-2)+3}{x^2 - 2x + 3}\, dx \\
&= \int \frac{2x-2}{x^2 - 2x + 3}\, dx + \int \frac{3}{x^2 - 2x + 3}\, dx \\
&= (① \ u\text{−substitution part}) + (② \text{ arctan part})
\end{aligned}$$

①

$$\begin{aligned}
\int \frac{2x-2}{x^2 - 2x + 3}\, dx = &= \int \frac{du}{u}\, dx \\
&\quad (u = x^2 - 2x + 3, du = (2x-2)dx) \\
&= \ln|u| + C \\
&= \ln|x^2 - 2x + 3| + C \\
&= \ln(x^2 - 2x + 3) + C.
\end{aligned}$$

② How about $\displaystyle\int \frac{3}{x^2 - 2x + 3}\, dx = 3 \int \frac{1}{x^2 - 2x + 3}\, dx$?

Let's compute $\displaystyle\int \frac{1}{x^2 - 2x + 3}\, dx$.

$$\begin{aligned}
\int \frac{1}{x^2 - 2x + 3}\, dx &= \int \frac{1}{(x-1)^2 + 2}\, dx \\
&\quad (u = x - 1, du = dx) \\
&= \int \frac{1}{u^2 + 2}\, du \\
&\quad (u = \sqrt{2} \tan\theta, du = \sqrt{2} \sec^2\theta d\theta) \\
&= \int \frac{\sqrt{2} \sec^2\theta d\theta}{2 \sec^2\theta} = \frac{\sqrt{2}}{2} 1\ d\theta \\
&= \frac{\sqrt{2}}{2}\theta + C = \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\
&= \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C.
\end{aligned}$$

Part III: Final Conclusion

$$\begin{aligned}
\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}\, dx &= \int \frac{5}{x-2}\, dx + \int \frac{2x+1}{x^2 - 2x + 3}\, dx \\
&= \int \frac{5}{x-2}\, dx + \int \frac{2x-2}{x^2 - 2x + 3}\, dx + \int \frac{3}{x^2 - 2x + 3}\, dx \\
&= 5 \ln|x - 2| + \ln(x^2 + 2x + 3) + 3\frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C.
\end{aligned}$$

Lesson 18

Topics: Improper Integrals
Section Number: 8.9
Lecture Plan:

(1) Use the first 15 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 17. This should serve as a review for Lesson 17.
- Review for Lesson 17
- MyLabMath Homework for Lesson 17

(2) Explain the 3 Types of Improper Integral
- Type I: One of the integration limit is ∞ (or $-\infty$)
- Type II: The integrand is not defined at one of the integration limit
- Type III: The integrand is not defined at one of the points in the interval of integration

(3) Discussion of Type I
- Example Problems

Example Problem 1: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx := \lim_{b\to\infty}\int \frac{1}{x^2}\,dx.$

Example Problem 2:

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}\,dx &:= \lim_{b\to\infty}\int \frac{1}{1+x^2}\,dx\\ &= \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = \pi/2\end{aligned}$$

Note: Explain the last step of the computation using a picture.

Example Problem 3: Analyze the behavior of

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\frac{1}{x^p}\,dx$$

depending on the value of p.

(4) Discussion of Type II
- Example Problems

Example Problem 3:

$$\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx := \lim_{b\to 3^-}\int \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{b\to 3^-}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b = \pi/2$$

Note:

◦ Explaination of $\lim_{b\to 3^-}$ (why approaching from the negative side) using a picture

◦ Explaination of the last step of the computation using a picture

(5) Discussion of Type III

Example Problem 4:

$$\begin{aligned}\int_1^3 \frac{1}{(x-2)^{1/3}}\,dx &:= \lim_{c\to 2^-}\int_1^c \frac{1}{(x-2)^{1/3}}\,dx + \lim_{d\to 2^+}\int_d^3 \frac{1}{(x-2)^{1/3}}\,dx\\ &= -3/2 + 6\\ &= 9/2\end{aligned}$$

Note:

◦ Explaination of $\lim_{c\to 2^-}$ (why approaching from the negative side) and $\lim_{d\to 2^+}$ (why approaching from the positive side) using a picture

(6) Common Mistake

- Problem: Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}\ dx$ coverges or diverges. If it converges, comoute its value.

◦ Correct Answer:

$$\begin{array}{rcccc} \int_{-\infty}^{\infty} xe^{-x^2}\ dx & := & \lim_{c\to-\infty} \int_c^0 xe^{-x^2}\ dx & + & \lim_{d\to\infty} \int_0^\infty xe^{-x^2}\ dx \\ & = & \lim_{c\to-\infty} \left[-\frac{1}{2}e^{-x^2}\right]_c^0 xe^{-x^2}\ dx & + & \lim_{d\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_0^d \\ & = & -1/2 & + & 1/2 \\ & = & 0 & & \end{array}$$

Therefore, the improper integral coverges to the value 0.

◦ Fake Answer:

$$\int_{-\infty}^{\infty} xe^{-x^2}\ dx = \lim_{b\to\infty} \int_{-b}^{b} xe^{-x^2}\ dx = 0,$$

since xe^{-x^2} is an odd function. Therefore, the improper integral coverges to the value 0.

It is also brilliant, as it avoids the complicated computation as in the Correct Answer. We reach the same asnswer any way ! What matter is the final answer, ha, ha, ha !

Warning: This is a WRONG argument. We should NOT take the "combined" limit "$\lim_{b\to\infty} \int_{-b}^{b}$".

Consider the following example: $\int_{-\infty}^{\infty} x\ dx$

◦ Correct Answer:

$$\begin{array}{rcccc} \int_{-\infty}^{\infty} x\ dx & := & \lim_{c\to-\infty} \int_c^0 x\ dx & + & \lim_{d\to\infty} \int_0^\infty x\ dx \\ & = & \lim_{c\to-\infty} \left[\frac{1}{2}x^2\right]_c^0 & + & \lim_{d\to\infty} \left[\frac{1}{2}x^2\right]_0^d \\ & = & -\infty & + & \infty \end{array}$$

Therefore, the improper integral $\int_{-\infty}^{\infty} x\ dx$ diverges.

◦ Fake Answer:

$$\int_{-\infty}^{\infty} x\ dx = \lim_{b\to\infty} \int_{-b}^{b} x\ dx = 0.$$

Lesson 19

Topics: Sequence and its limit
Section Number: 10.1, 10.2
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 18. This should serve as a review for Lesson 18.
- Review for Lesson 18
- MyLabMath Homework for Lesson 18

(2) What is a seqeunce ?
- Answer: A bunch of numbers indexed by the natural numbers $n \in \mathbb{N}$.

Note: Iindexing most of the time starts from $n = 1$, but not always.
Notations: $\{a_1, a_2, a_3, \ldots\}$, $\{a_n\}_{n\in\mathbb{N}}$ etc.
- Examples

① $a_n = \dfrac{1}{2^n}$, $\left\{a_1 = \dfrac{1}{2}, a_2 = \dfrac{1}{2^2}, a_3 = \dfrac{1}{2^3}, \ldots\right\}$.

② $a_n = \dfrac{(-1)^n}{n^2+1}$, $\left\{a_1 = -\dfrac{1}{2}, a_2 = \dfrac{1}{5}, a_3 = -\dfrac{1}{10}, \ldots\right\}$.

③ a sequence defined by a recurrence relation

$$\begin{cases} a_1 & = & 1 \\ a_{n+1} & = & 2a_n + 1. \end{cases}$$

$$\begin{array}{lcl} a_1 & = & 1, \\ a_2 & = & 2a_1 + 1 = 3, \\ a_3 & = & 2a_2 + 1 = 7, \\ a_4 & = & 2a_3 + 1 = 15, \\ & \cdots & \end{array}$$

④ a geometric sequence defined by the recurrence realtion
$a_{n+1} = ra_n$ with r being some constant $\longrightarrow a_n = a \cdot r^{n-1}$ with $a = a_1$

(3) Limit of a sequence $\lim_{n\to\infty} a_n$
- Easy Examples

④ $a_n = \dfrac{(-1)^n}{n^2+1}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$$

since

$$\begin{array}{ccccccc} & \dfrac{-1}{n^2+1} & \le & a_n & \le & \dfrac{1}{n^2+1} \\ & & & \text{Squeeze Th.} & & \\ n \to \infty & \downarrow & & \downarrow & & \downarrow \\ & 0 & & 0 & & 0 \end{array}$$

⑤ $a_n = \cos(n\pi)$,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi)$ DNE

◦ Explain using a picture.

⑥

$$\begin{cases} a_1 & = & 1 \\ a_{n+1} & = & -2a_n \end{cases} \longrightarrow a_n = (-2)^{n-1}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^{n-1}$ DNE

⑦ $a_n = \dfrac{4n^3}{n^3+1}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^3}{n^3+1} = 4$$

• Difficult Examples

⑦ $a_n = \left(\dfrac{n+5}{n}\right)^n$,

Exercise from MA 165: Show $\lim_{x\to\infty} \left(\dfrac{x+5}{x}\right)^x = e^5$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n = e^5$$

⑧ $a_n = n^{1/n}$,

Exercise from MA 165: Show $\lim_{x\to\infty} x^{1/x} = 1$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{1/n} = 1$$

• Limit of a geometric sequence $a_n = a \cdot r^{n-1}$ with $a = a_1 \neq 0$

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if} \quad |r| < 1 \\ a & \text{if} \quad r = 1 \\ \text{diverges} & \text{otherwise} \end{cases}$$

(4) Monotone Sequence Theorem

• Statement:

A monotone bounded sequence converges to some finite number !

That is to say, if $\{a_n\}$ is a monotone bounded sequqnce, then

$$\lim_{n\to\infty} a_n = L \quad \text{(exists as a finite number)}$$

• Explanation of the key words

∘ what is a monotone sequence ?

∘ what is the meaning of "a sequence being bounded" ?

• FAQ: Isn't it an abstract existence theorem, which is useless to compute the actual limit ?

Answer: NO.

Example Problem: A sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation

$$\begin{cases} a_1 &= 100 \\ a_{n+1} &= \dfrac{1}{2}a_n + 100. \end{cases}$$

Find $\lim_{n\to\infty} a_n$.

Solution:

Step 1: Check the following two conditions by **mathematical induction**.

① The seqeuence is monotone.

② The seqeuence is bounded.

Step 2. Using the Monotone Sequence Theorem, we compute the limit.

Recurrence relation

$$a_{n+1} = \frac{1}{2}a_n + 100$$

$\longrightarrow$

$$\begin{array}{ccc} \lim_{n\to\infty} (a_{n+1}) & = & \lim_{n\to\infty} \left(\dfrac{1}{2}a_n + 100\right) \\ \| & & \| \\ L & & \dfrac{1}{2}L + 100 \end{array}$$

$\longrightarrow$

$$\lim_{n\to\infty} a_n = L = 200.$$

Lesson 20

Topics: Introduction of the Series (Telescoping Series)
Section Number: 10.1, 10.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 19. This should serve as a review for Lesson 19.
  - Review for Lesson 19
  - MyLabMath Homework for Lesson 19

(2) What is a series ?
Answer: Given a sequence $\{a_k\}_{k=1}^{\infty}$,
the series is the sum of all the terms $\sum_{k=1}^{\infty} a_k$!
That is to say,
given $\{a_k\}_{k=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$,
we add all the terms to get the series
$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdot$.

WARNING: Since we can NOT add infinitely many time, we have to make it clear the meaning of $\sum_{k=1}^{\infty} a_k$.

Definition: Given a sequence, we define the partial sums

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\cdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

Then we define

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n.$$

Terminology: Convergence and **Divergence**

$$\text{The series is} \begin{cases} \textbf{convergent} \\ \textbf{divergent} \end{cases} \text{if} \begin{cases} \lim_{n\to\infty} S_n \ \text{ exists and finite} \\ \lim_{n\to\infty} S_n = \pm\infty \ \text{ or DNE} \end{cases}$$

(3) Telescoping Series
Example Problem 1: Given a sequence $\{a_k\}_{k=1}^{\infty}$ where $a_k = \dfrac{1}{k(k+1)}$,
compute the series $\sum_{k=1}^{\infty} a_k$.
Solution:
Observe

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

We compute

$$\begin{aligned} S_n &= && a_1 \\ & + && a_2 \\ & + && a_3 \\ & \cdots && \cdots \\ & + && a_{n-1} \\ & + && a_n \\ &= && \frac{1}{1} - \cancel{\frac{1}{2}} \\ & + && \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \\ & + && \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \\ & \cdots && \cdots \\ & + && \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} \\ & + && \cancel{\frac{1}{n}} - \frac{1}{n+1} \end{aligned}$$

and hence

$$S_n = \frac{1}{1} - \frac{1}{n+1}.$$

Therefore, we conclude

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1} \right) = 1.$$

Example Problem 2: Given a sequence $\{a_k\}_{k=1}^{\infty}$ where $a_k = \dfrac{1}{9k^2+15k+4}$, compute the series $\sum_{k=1}^{\infty} a_k$.

Solution:

Observe

$$\begin{aligned} a_k &= \frac{1}{9k^2+15k+4} \\ &= \frac{1}{3}\left\{ \frac{1}{3k+1} - \frac{1}{3k+4} \right\} \\ &= \frac{1}{3}\left\{ \frac{1}{3k+1} - \frac{1}{3(k+1)+1} \right\} \end{aligned}$$

We compute

$$\begin{aligned} S_n &= && a_1 \\ & + && a_2 \\ & + && a_3 \\ & \cdots && \cdots \\ & + && a_{n-1} \\ & + && a_n \\ &= && \frac{1}{3}\left\{ \frac{1}{3\cdot 1+1} - \cancel{\frac{1}{3\cdot 2+1}} \right\} \\ & + && \frac{1}{3}\left\{ \cancel{\frac{1}{3\cdot 2+1}} - \cancel{\frac{1}{3\cdot 3+1}} \right\} \\ & + && \frac{1}{3}\left\{ \cancel{\frac{1}{3\cdot 3+1}} - \cancel{\frac{1}{3\cdot 4+1}} \right\} \\ & \cdots && \cdots \\ & + && \frac{1}{3}\left\{ \cancel{\frac{1}{3(n-1)+1}} - \cancel{\frac{1}{3n+1}} \right\} \\ & + && \frac{1}{3}\left\{ \cancel{\frac{1}{3n+1}} - \frac{1}{3(n+1)+1} \right\} \end{aligned}$$

and hence

$$S_n = \frac{1}{3}\left\{\frac{1}{3\cdot 1+1} - \frac{1}{3(n+1)+1}\right\}$$

Therefore, we conclude

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{3}\left\{\frac{1}{3\cdot 1+1} - \frac{1}{3(n+1)+1}\right\} = \frac{1}{12}.$$

(4) Geometric Series

Problem: $\sum_{k=\alpha}^{\infty} a_k =?$ where $a_{k+1} = r\cdot a_k$ for some constant r.

Note:

① The starting number α may not be 1.)

② We assume $a_\alpha = a \neq 0$. If $a = 0$, then $\sum_{k=\alpha}^{\infty} a_k = 0$.

Solution:

$$\begin{array}{rclcccccccc}
S_n & = & a_\alpha & + & a_{\alpha+1} & + & \cdots & + & a_n & & \\
 & = & a & + & a\cdot r & + & \cdots & + & a\cdot r^{n-\alpha} & & \\
 & & & \searrow & & \searrow & \cdots & \searrow & & \searrow & \\
-\ )\ \ rS_n & = & & & a\cdot r & + & \cdots & + & a\cdot r^{n-\alpha} & + & a\cdot r^{n-\alpha+1} \\
\hline
(1-r)S_n & = & a & & & & & & & - & a\cdot r^{n-\alpha+1} \\
 & = & \multicolumn{9}{l}{a(1-r^{n-\alpha+1})}
\end{array}$$

$\longrightarrow$

$$S_n = \begin{cases} \dfrac{a(1-r^{n-\alpha+1})}{1-r} & \text{when} \quad r\neq 1 \\ na & \text{when} \quad r = 1 \end{cases}$$

$\longrightarrow$

$$\sum_{k=\alpha}^{\infty} a_k = \lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r} & \text{if} \quad |r|<1 \\ \text{diverges} & \text{otherwise} \end{cases}$$

Example Problem: Dtermine whether $\sum_{k=3}^{\infty} a_k$ converges or diverges where

$$a_k = \frac{19\cdot 3^{4k+2}}{7^{5k-1}}.$$

Solution:

$$\frac{a_{k+1}}{a_k} = \frac{\dfrac{19\cdot 3^{4(k+1)+2}}{7^{5(k+1)-1}}}{\dfrac{19\cdot 3^{4k+2}}{7^{5k-1}}} = \frac{3^4}{7^5} = r \text{ is a constant.}$$

$\longrightarrow$

$\{a_k\}$ is a geometric sequence with $r = \dfrac{3^4}{7^5}$.

Now

$$\begin{cases} a & = & a_3 = \dfrac{19\cdot 3^{4\cdot 3+2}}{7^{5\cdot 3-1}} \\ r & = & \dfrac{3^4}{7^5} \quad \text{with} |r| < 1. \end{cases}$$

$\longrightarrow$

$\sum_{k=3}^{\infty} a_k$ converges to $\dfrac{a}{1-r} = \dfrac{\dfrac{19\cdot 3^{4\cdot 3+2}}{7^{5\cdot 3-1}}}{1-\dfrac{3^4}{7^5}}$.

Lesson 21

Topics: Divergence Test
Section Number: 10.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 20. This should serve as a review for Lesson 20.
- Review for Lesson 20
- MyLabMath Homework for Lesson 20

(2) Divergence Test
- Statement: $\lim_{k\to\infty} a_k \neq 0 \Longrightarrow \sum_{k=\text{wahtever}}^{\infty} a_k$ diverges.

Strong WARNING: Do NOT make up your FALSE "Convergence Test":
The statement ($\lim_{k\to\infty} a_k = 0 \Longrightarrow \sum_{k=\text{whatever}}^{\infty} a_k$ converges) is FALSE.

- Examples

① $\sum_{k=0}^{\infty} \dfrac{k}{k+1} = \sum_{k=0}^{\infty} a_k$ with $a_k = \dfrac{k}{k+1}$.

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{k}{k+1} = 1 \neq 0 \Longrightarrow \sum_{k=\text{wahtever}}^{\infty} a_k$ diverges.

② $\sum_{k=1}^{\infty} \dfrac{1+3^k}{2^k} = \sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1+3^k}{2^k}$.

$$\begin{array}{cccccccc} & a_k & = & \dfrac{1+3^k}{2^k} & > & \dfrac{3^k}{2^k} & = & \left(\dfrac{3}{2}\right)^k \\ k\to\infty & & & \downarrow & & & & \downarrow \\ & & & \infty & & & & \infty \end{array}$$

$\lim_{k\to\infty} a_k \infty \neq 0 \Longrightarrow \sum_{k=\text{wahtever}}^{\infty} a_k$ diverges.

③ $\sum_{k=1}^{\infty} \dfrac{1}{k} = \sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1}{k}$.

We compute $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{1}{k} = 0$.

This means that Divergence Test is INCONCLUSIVE.

Note: Actually $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges (as we will see using the Integral Test).

④ $\sum_{k=1}^{\infty} \dfrac{1}{k^2} = \sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1}{k^2}$.

We compute $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{1}{k^2} = 0$.

This means that Divergence Test is INCONCLUSIVE.

Note: Actually $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ converges (as we will see using the Integral Test).

(3) Proof of Divergence Test (Optional: HARD for the students since it uses LOGIC.)
- Explantion of "contrapositive"
  - $A \Longrightarrow B$ is equivalent to (not B) $\Longrightarrow$ (not A)
- Our case
  - A: $\lim_{k\to\infty} a_k \infty \neq 0$
  - B: $\sum_{k=\text{whatever}}^{\infty} a_k$ diverges.
  - (not A): $\lim_{k\to\infty} a_k = 0$
  - (not B): $\sum_{k=\text{whatever}}^{\infty} a_k$ converges.

- Enought to show

$\sum_{k=\text{whatever}}^{\infty} a_k$ converges. $\Longrightarrow \lim_{k\to\infty} a_k = 0$

(Proof)

Suppose $\sum_{k=\text{whatever}}^{\infty} a_k$ converges.

$\longrightarrow$

$\lim_{n\to\infty} S_n = L$ exists where L is some finte number.

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \lim_{k\to\infty}(S_k - S_{k-1}) \\
&= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\
&= L - L = 0.
\end{aligned}$$

Lesson 22

Topics: Integral Test and p-Series
Section Number: 10.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 21. This should serve as a review for Lesson 21.
- Review for Lesson 21
- MyLabMath Homework for Lesson 21

(2) Integral Test
- Explain the main idea in the case of the harmonic series using a picture
- Statement: Suppose we are given $\{a_k\}_{k=1}^{\infty}$ a sequence.

Suppose we have a function $f(x)$ defined over $[1,\infty)$ satisfying the following conditiions over $[1,\infty)$:

⓪ $a_k = f(k)$,
① continuous,
② positive,
③ decreasing (in the sense that $f(x) \le f(x')$ if $x \le x'$).

Then $\sum_{k=1}^{\infty} a_k$ and $\displaystyle\int_1^{\infty} f(x)\ dx$ shares the SAME destiny.

That is to say,

$$\left\{\begin{array}{ccc} \sum_{k=1}^{\infty} a_k \text{ converges} & \Longleftrightarrow & \displaystyle\int_1^{\infty} f(x)\ dx\text{converges} \\ \sum_{k=1}^{\infty} a_k \text{ diverges} & \Longleftrightarrow & \displaystyle\int_1^{\infty} f(x)\ dx\text{diverges} \end{array}\right\}$$

Note:

(i) When they converge, the value of $\sum_{k=1}^{\infty} a_k$ may not be equal to that of $\displaystyle\int_1^{\infty} f(x)\ dx$.

(ii) The starting number for the series may not be equal to 1.

(iii) The conditions for the function to satisfy should only be checked for the inetrval $[\alpha,\infty)$ for some α, and we only have to compute $\displaystyle\int_\alpha^{\infty} f(x)\ dx$ accordingly.

- Example Problems

Example Problem 1: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1}{k^2+1}$ converges or diverges, using the Integral Test.

Solution: Set $f(x) = \dfrac{1}{x^2+1}$.

Then $f(x)$ satisfies (over $[1,\infty)$)

⓪ $a_k = f(k)$? Yes, obvious ✓
① continuous ? Yes, obvious ✓
② positive ? Yes, obvious ✓
③ decreasing ? Yes, since $f'(x) = \dfrac{-2x}{(x^2+1)^2} < 0$

We compute

$$\begin{array}{rcl} \displaystyle\int_1^{\infty} f(x)\ dx & = & \lim_{b\to\infty} \displaystyle\int_1^{b} f(x)\ dx \\ & = & \lim_{b\to\infty} \displaystyle\int_1^{b} \frac{1}{x^2+1}\ dx \\ & = & \lim_{b\to\infty} \left[\tan^{-1} x\right]_1^b \\ & = & \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \pi/2 - \pi/4 = \pi/4. \end{array}$$

$\xrightarrow{\text{Integral Test}} \sum_{k=1}^{\infty} a_k$ converges.

Note:
(i) Explain why $\lim_{b\to\infty}\tan^{-1} b = \pi/2$, using a picture.
(ii) Actually $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \dfrac{1}{k^2+1} > \displaystyle\int_1^{\infty} f(x)\ dx$.

Example Problem 2: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{k}{k^2+1}$ converges or diverges, using the Integral Test.
Solution: Set $f(x) = \dfrac{x}{x^2+1}$.
Then $f(x)$ satisfies (over $[2,\infty)$)

⓪ $a_k = f(k)$? Yes, obvious ✓
① continuous ? Yes, obvious ✓
② positive ? Yes, obvious ✓
③ decreasing ? Yes, since $f'(x) = \dfrac{-x^2+1}{(x^2+1)^2} < 0$

We compute

$$
\begin{aligned}
\int_2^{\infty} f(x)\ dx &= \lim_{b\to\infty} \int_2^b f(x)\ dx \\
&= \lim_{b\to\infty} \int_2^b \frac{x}{x^2+1}\ dx \\
&\quad \left(\begin{array}{lll} x & u = x^2+1 & du = 2x\ dx \\ b & b^2+1 & \\ 2 & 5 & \end{array}\right) \\
&= \lim_{b\to\infty} \int_5^{b^2+1} \frac{1}{u}\left(\frac{1}{2}du\right) \\
&= \lim_{b\to\infty} \frac{1}{2}\left[\ln u\right]_5^{b^2+1} \\
&= \lim_{b\to\infty} \frac{1}{2}\left[\ln(b^2+1) - \ln 5\right] = \infty.
\end{aligned}
$$

$\xrightarrow{\text{Integral Test}} \sum_{k=1}^{\infty} a_k$ diverges.

(3) p-Series

- What is a p-series ?

Answer: It is a series of the form $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$, where p is some fixed number.

- Statement: The p-series converges for $p > 1$, and diverges for $p \leq 1$.

That is to say,

$$\text{the } p\text{-series is } \begin{cases} \text{convergent} \\ \text{divergent} \end{cases} \text{if } \begin{cases} p > 1 \\ p \leq 1 \end{cases}$$

Proof:

∘ $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$ converges for $p > 1 \longleftarrow$ Use Integral Test.
∘ $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$ diverges for $0 \leq p \leq 1 \longleftarrow$ Use Integral Test.
∘ $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$ diverges for $p < 0 \longleftarrow$ Use Divergence Test.

- Examples

∘ $\sum_{k=1}^{\infty} \dfrac{1}{\sqrt{k}}$ a p-series with $p = 1/2 \leq 1 \longrightarrow$ diverges.
∘ $\sum_{k=1}^{\infty} \dfrac{1}{k}$ a p-series with $p = 1 \leq 1$ (harmonic series) $\longrightarrow$ diverges.
∘ $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ a p-series with $p = 2 > 1 \longrightarrow$ converges.
∘ $\sum_{k=1}^{\infty} k$ a p-series with $p = -1 \leq 1 \longrightarrow$ diverges.

Lesson 23

Topics: Comparison Test & Limit Comparison Test
Section Number: 10.5
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 22. This should serve as a review for Lesson 22.
- Review for Lesson 22
- MyLabMath Homework for Lesson 22

(2) Review on Geometric Series and p-Series
- Geometric Series with initial term $a \neq 0$ and ratio r s.t. $a_{k+1} = r \cdot a_k$
  - $\circ \sum a_k = \begin{cases} \text{converges to } \dfrac{a}{1-r} & \text{if} \quad |r| < 1 \\ \text{diverges} & \text{if} \quad |r| \geq 1 \end{cases}$
- p-series with $a_k = \dfrac{1}{k^p}$
  - $\circ \sum a_k = \begin{cases} \text{converges} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \leq 1 \end{cases}$

(3) Comparison Test
- Statement:

$\{a_k\}, \{b_k\}$ two sequences with $a_k, b_k \geq 0$
Case: $a_k \leq b_k$: $\sum a_k$ converges $\Longleftarrow \sum b_k$ converges.
Case: $b_k \leq a_k$: $\sum b_k$ diverges $\Longrightarrow \sum a_k$ diverges.
Note: In practice,
given a sequence $\{a_k\}$, you try to find another sequence $\{b_k\}$
which satisfies the required inequality, and
which is simpler and hence you know whose convergence (or divergence).

- Example Probelms

Example Problem 1: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1}{k^2+10}$ converges or diverges, using the Comparison Test.

Solution.
Set $b_k = \dfrac{1}{k^2}$.
Then we have $a_k = \dfrac{1}{k^2+10} \leq \dfrac{1}{k^2} = b_k$.
Observe $\sum b_k = \sum \dfrac{1}{k^2}$ converges, since it is a p-series with $p = 2 > 1$.
$\Longrightarrow$
$\sum_{k=1}^{\infty} a_k$ converges.

Example Problem 2: Determine whether the series $\sum_{k=4}^{\infty} a_k$ with $a_k = \dfrac{1}{\sqrt{4-3}}$ converges or diverges, using the Comparison Test.

Solution.
Set $b_k = \dfrac{1}{\sqrt{k}}$.
Then we have $b_k = \dfrac{1}{\sqrt{k}} \leq \dfrac{1}{\sqrt{4-3}} = a_k$.
Observe $\sum b_k = \sum \dfrac{1}{\sqrt{k}} = \sum \dfrac{1}{k^{1/2}}$ diverges, since it is a p-series with $p = 1/2 \leq$.
$\Longrightarrow$
$\sum_{k=1}^{\infty} a_k$ diverges.

(4) Limit Comparision Test

• Statement:

$\{a_k\}, \{b_k\}$ two sequences with $a_k, b_k > 0$

$\lim_{k\to\infty} \dfrac{a_k}{b_k} = L \neq 0 \Longrightarrow \sum a_k$ and $\sum b)k$ shares the SAME destiny.

That is to say,

$$\left\{ \begin{array}{ccc} \sum a_k \text{ converges} & \Longleftrightarrow & \sum b_k \text{converges} \\ \sum a_k \text{ diverges} & \Longleftrightarrow & \sum b_k \text{diverges} \end{array} \right\}$$

Note: In practice,

given a sequence $\{a_k\}$, you try to find another sequence $\{b_k\}$

which is similar to a_k in the sense $\lim_{k\to\infty} \dfrac{a_k}{b_k} = L \neq 0$,

which is yet simpler and hence you know whose convergence (or divergence).

• Variants of L.C.M.

(i) $\lim_{k\to\infty} \dfrac{a_k}{b_k} = L = 0$ & $\sum b_k$ converges $\Longrightarrow \sum a_k$ converges.

(ii) $\lim_{k\to\infty} \dfrac{a_k}{b_k} = \infty$ & $\sum b_k$ diverges $\Longrightarrow \sum a_k$ diverges.

• Example Problems

Example Problem 3: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{5k^4 - 2k^2 + 3}{2k^2 - k + 5}$ converges or diverges, using the Limit Comparison Test.

Solution.

Set $b_k = \dfrac{k^4}{k^6} = \dfrac{1}{k^2}$.

Then $\lim_{k\to\infty} \dfrac{a_k}{b_k} = \dfrac{5}{2} = L \neq 0$.

$\overset{\text{L.C.T.}}{\Longrightarrow}$

$\sum a_k$ and $\sum b)k$ shares the SAME destiny.

&

$\sum b)k = \sum \dfrac{1}{k^2}$ converges, since it is a p-series with $p = 2 > 1$.

$\longrightarrow$

$\sum a_k$ converges.

Example Problem 4: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{\ln k}{k^2}$ converges or diverges, using the Limit Comparison Test.

Solution.

<u>1st Attempt</u>

Set $b_k = \dfrac{1}{k^2}$.

Then $\lim_{k\to\infty} \dfrac{a_k}{b_k} = \lim_{k\to\infty} \dfrac{\ln k/k^2}{1/k^2} = \lim_{k\to\infty} \ln k = \infty$.

On the other hand, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \dfrac{1}{k^2}$ converges, since it is a p-series with $p = 2 > 1$.

We can NOT use L.C.M. or variant (ii) of L.C.M.

<u>2nd Attempt</u>

Set $b_k = \dfrac{1}{k}$.

Then $\lim_{k\to\infty} \dfrac{a_k}{b_k} = \lim_{k\to\infty} \dfrac{\ln k/k^2}{1/k} = \lim_{k\to\infty} \dfrac{\ln k}{k} = 0$.

(Note: $\lim_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{L'Hospital's Rule}}{=} \lim_{x\to\infty} \dfrac{1/x}{1} = 0$.)

On the other hand, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, since it is the harmonic series.

We can NOT use L.C.M. or variant (i) of L.C.M.

3rd Attempt

Set $b_k = \frac{1}{k^{3/2}}$.

(Note: $\frac{1}{k^2}$(used in 1st attempt) $< b_k = \frac{1}{k^{3/2}} < \frac{1}{k}$(used in 2nd attempt).)

Then $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k^{3/2}} = \lim_{k\to\infty} \frac{\ln k}{k^{1/2}} = 0$.

(Note: $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} \overset{\text{L'Hospital's Rule}}{=} \lim_{x\to\infty} \frac{1/x}{x^{-1/2}/2} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$.)

Moreover, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, since it is a p-series with $p = 3/2 > 1$.

$\overset{\text{variant (i) of L.C.M.}}{\longrightarrow}$ $\sum_{k=1}^{\infty} a_k$ converges.

Example Problem 5 (Optional): Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{\ln k}{k^2}$ converges or diverges, using the Integral Test.

Solution.

Set $f(x) = \frac{\ln x}{x^2}$.

Then $f(x)$ satisfies (over $[2, \infty)$)

- ⓪ $a_k = f(k)$? Yes, obvious ✓
- ① continuous ? Yes, obvious ✓
- ② positive ? Yes, obvious ✓
- ③ decreasing ? Yes, since $f'(x) = \frac{x(1 - 2\ln x)}{x^4} < 0$.

We compute

$$
\begin{aligned}
\int_1^{\infty} f(x)\ dx &= \lim_{b\to\infty} \int_1^b f(x)\ dx \\
&= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\ dx \\
& \quad \left(\begin{array}{ccccccc} u & = & \ln x & , & v & = & -\frac{1}{x} \\ du & = & \frac{1}{x}dx & , & dv & = & \frac{1}{x^2}dx \end{array} \right) \\
&= \lim_{b\to\infty} \left([uv]_1^b - \int_1^b v du \right) \\
&= \lim_{b\to\infty} \left(\left[\ln x \cdot \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \cdot \frac{1}{x} dx \right) \\
&= \lim_{b\to\infty} \left(\left[\ln x \cdot \left(-\frac{1}{x} \right) \right]_1^b + \int_1^b \frac{1}{x^2} dx \right) \\
&= \lim_{b\to\infty} \left(\left[\ln x \cdot \left(-\frac{1}{x} \right) \right]_1^b + \left[-\frac{1}{x} \right]_1^b \right) \\
&= \lim_{b\to\infty} \left(\left[-\frac{\ln b}{b} \right] + \left[\left(-\frac{1}{b} \right) - \left(-\frac{1}{1} \right) \right] \right) \\
&= 0 + 1 = 1
\end{aligned}
$$

$\overset{\text{Integral Test}}{\longrightarrow}$ $\sum_{k=1}^{\infty} a_k$ converges.

Lesson 24

Topics: Alternaating Series Test & Absolute/Conditional Convergence
Section Number: 10.6
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 23. This should serve as a review for Lesson 23.
- Review for Lesson 23
- MyLabMath Homework for Lesson 23

(2) What is an alternating series ?
Answer: It is the sum of a sequence whose terms alter their signs.
It is either of the form $\sum(-1)^{k+1}b_k$ or $\sum(-1)^k b_k \quad (b_k > 0)$.
- Examples
 - $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}b_k$ with $b_k = \frac{1}{k}$
 - $-\frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots = \sum_{k=1}^{\infty}(-1)^k b_k$ with $b_k = \frac{1}{k+2}$

(3) Alternating Series Test
- Statement:
An alternating series $\sum(-1)^{k+1}b_k \quad (b_k > 0)$ (or $\sum(-1)^k b_k \quad (b_k > 0)$) converges if the following two conditions are satisfied:
Condition ① b_k is decreasing (in the sense $b_k \geq b_{k+1}$),
Condition ② $\lim_{k\to\infty} b_k = 0$.

Note:
(i) The failure of condition ② implies that the alternating series diverges by Divergence Test.
(ii) The failure of condition ① by itself does NOT guarantee that the alternating series diverges.

(4) Examples

Example 1: Consider the alternating series $\sum_k^{\infty}(-1)^{k+1}b_k$ with $b_k = \frac{1}{k}$.
Then we check
Condition ① b_k is decreasing ?
Yes, because $b_k = \frac{1}{k} \geq \frac{1}{k+1} = b_{k+1}$ ✓
Condition ② $\lim_{k\to\infty} b_k = 0$?
Yes, because $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{k} = 0.$ ✓
$\xrightarrow{\text{A.S.T.}}$
The alternating series $\sum_1^{\infty}(-1)^{k+1}b_k$ converges.
Note: Actually we have $\sum(-1)^{k+1}b_k = \sum(-1)^{k+1}\frac{1}{k} = \ln 2$.
Isn't it amazing !

Example 2: Consider the alternating series $\sum_1^{\infty}(-1)^{k+1}b_k$ with $b_k = \frac{k+1}{k}$.
Then we check
Condition ① b_k is decreasing ?
Yes, because $b_k = \frac{k+1}{k} \geq \frac{k+2}{k+1} = b_{k+1}$ ✓
Note: $\frac{k+1}{k} - \frac{k+2}{k+1} = \frac{1}{k(k+1)} > 0$.
Condition ② $\lim_{k\to\infty} b_k = 0$?
No, because $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k+1}{k} = 1 \neq 0$.
$\longrightarrow$
$\lim_{k\to\infty} a_k$ DNE where $a_k = (-1)^{k+1}b_k$.

$\xrightarrow{\text{Divergence Test}}$

The alternating series $\sum_1^\infty a_k = \sum_1^\infty (-1)^{k+1} b_k$ diverges.

(5) Estimation Theorem for Alternating Series

• Statement: Suppose we have an alternating series $\sum(-1)^{k+1}b_k \quad (b_k > 0)$ (or $\sum(-1)^k b_k \quad (b_k > 0)$) satisfying the following two conditions:

Condition ① b_k is decreasing (in the sense $b_k \geq b_{k+1}$),

Condition ② $\lim_{k\to\infty} b_k = 0$.

Then A.S.T. says it converges to a finite number, i.e., $\sum(-1)^{k+1}b_k = S$.

We have the estimate

$$|S - S_n| \leq b_{n+1}.$$

• Explanation using a picture

• Example Problem: Mean Boss (whose name is Kenji Matsuki) tells you to compute the value of the alternating series

$$\sum_{k=1}^{\infty}(-1)^k b_k = \sum_{k=1}^{\infty}(-1)^k \frac{1}{k!} = -\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \quad \text{with } b_k = \frac{1}{k!},$$

since he knows it satisfies conditions ① and ② stated in A.S.T. and hence that it converges to a finte value S. When you refuse to compute, saying it is an impossible task to add and subtract infinitely many times, he goes "O.K. I'll give you a break. You don't have to add and subtract infinitely many times. Just compute the partil sum S_n (adding from $k = 1$ to $k = n$). As long as the error is smaller than $\frac{1}{1000}$, i.e.,

$$|S - S_n| < \frac{1}{1000},$$

you can go back home."

You want to finish the job as soon as possible, and hence you want to make n (the number of terms to add and subtract) as small as possible. But you want to make sure that the error requirement is satsisfied in order for you not to be fired.

What is the smallest n that gurantees

$$|S - S_n| < \frac{1}{1000}$$

via the Estimation Theorem ?

Solution.

The Estimation Theorem says

$$|S - S_n| \leq b_{n+1}.$$

Therefore, it is enough to have

$$b_{n+1} = \frac{1}{(n+1)!} < \frac{1}{1000},$$

which is equivalent to

$$(n+1)! > 1000.$$

We construc the table

n	n+1	(n + 1)!	
1	2	2	
2	3	6	
3	4	24	
4	5	120	
5	6	720	< 1000
6	7	5040	> 1000

Therefore, the smallest n which satisfies $b_{n+1} < \frac{1}{1000}$ is $n = 6$.

(6) Absolute Convergence Test

- Statement: $\sum |a_k|$ converges $\Longrightarrow \sum a_k$ converges.
- Absolute Convergence vs Conditional Convergence

$\sum \lvert a_k \rvert$	$\sum a_k$	
convergent	convergent	absolutely convergent
divergent	convergent	conditionally convergent
convergent	divergent	N/A (cannot happen)
divergent	divergent	may happen, just say divergent

- Examples

$\circ$ $\sum_{k=1}^{\infty} a_k$ with $a_k = (-1)^{k+1}\dfrac{1}{k}$

$$\left\{\begin{array}{ccccc} \sum |a_k| & = & \sum \dfrac{1}{k} & \text{diverges} & \longleftarrow & \text{harmonic series} \\ \sum a_k & = & \sum \dfrac{1}{k} & \text{converges} & \longleftarrow & \text{A.S.T.} \end{array}\right.$$

$\longrightarrow$

$\sum a_k$ conditionally convergent.

$\circ$ $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{\sin k}{k^2}$

$\sum |a_k| = \sum |\dfrac{\sin k}{k^2}|$ is convergent $\xleftarrow{\text{ComparisonTest}}$ (we have $|\dfrac{\sin k}{k^2}| \le \dfrac{1}{k^2} = b_k$ and $\sum b_k$ converges.)

$\xrightarrow{\text{A.C.T.}}$

$\sum a_k$ absolutely convergent.

Lesson 25

Topics: Ratio Test & Root Test
Section Number: 10.7
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 24. This should serve as a review for Lesson 24.
- Review for Lesson 24
- MyLabMath Homework for Lesson 24

(2) Ratio Test
- Statement: Given a sequence $\{a_k\}$,

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}=\begin{cases} r<1 & \sum a_k \quad \text{(abs.) converges} \\ r=1 & \text{inconclusive} \\ r>1 \text{ (including the case } =\infty) & \sum a_k \quad \text{diverges} \\ \text{D.N.E.} & \text{inconclusive}\end{cases}$$

◦ Explanation of the rough idea: The series acts "almost" like a geometric sequence with ration r.

Note: It is absolutely crucial to take the absolute values of the terms before taking the ratio.

- Example Problems

Example Problem 1: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k=\dfrac{10^k}{k!}$ converges or diverges, using the Ratio Test.
Solution.

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}=\lim_{k\to\infty}\frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}}=\lim_{k\to\infty}\frac{10}{k+1}=0$$

$\xrightarrow{\text{Ratio Test}}$ $\sum_{k=1}^{\infty} a_k$ absolutely converges.

Example Problem 2: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k=\dfrac{(-1)^k k^k}{k!}$ converges or diverges, using the Ratio Test.
Solution.

$$\begin{aligned}\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty}\frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}}=\lim_{k\to\infty}\frac{\frac{(k+1)^k\cancel{(k+1)}}{\cancel{k!}\cancel{(k+1)}}}{\frac{k^k}{\cancel{k!}}} \\ &= \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k=e>1.\end{aligned}$$

$\xrightarrow{\text{Ratio Test}}$ $\sum_{k=1}^{\infty} a_k$ diverges.

(3) Crucial Limit Computation (Optional): $\lim_{k\to\infty}\left(1+\dfrac{b}{k}\right)^k=e^b$

We compute $\lim_{x\to\infty}\left(1+\dfrac{b}{x}\right)^x$ (formally 1^∞-form).

Set $y=\left(1+\dfrac{b}{x}\right)^x$.

Then $\ln y = x\ln\left(1+\dfrac{b}{x}\right)$.

We compute

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} x \ln\left(1+\frac{b}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{b}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'Hospital's rule}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{b}{x}}\left(b\cdot \cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = b.
\end{aligned}$$

$\longrightarrow$

$\lim_{x\to\infty} \ln y = b$

$\longrightarrow$

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^b$

$\longrightarrow$

$\lim_{k\to\infty} \left(1+\dfrac{b}{k}\right)^k = e^b.$

(4) Root Test

- Statement: Given a sequence $\{a_k\}$,

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \begin{cases} \rho < 1 & \sum a_k \quad \text{(abs.) converges} \\ \rho = 1 & \text{inconclusive} \\ \rho > 1 \text{ (including the case } = \infty) & \sum a_k \quad \text{diverges} \\ \text{D.N.E.} & \text{inconclusive} \end{cases}$$

Note: The statement as well as the basic idea is almost identical to that of the Ratio Test.

- Example Problems

Example Problem 1: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \left(\dfrac{3-4k^2}{7k^2+6}\right)^k$ converges or diverges, using the Root Test.
Solution.

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{|a_k|} &= \lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{3-4k^2}{7k^2+6}\right)^k\right|} \overset{3-4k^2<0}{=} \lim_{k\to\infty} \sqrt[k]{\left(\frac{4k^2-3}{7k^2+6}\right)^k} \\
&= \lim_{k\to\infty} \frac{4k^2-3}{7k^2+6} = \frac{4}{7} < 1.
\end{aligned}$$

$\xrightarrow{\text{Root Test}}$

$\sum_{k=1}^{\infty} a_k$ absolutely converges.

Example Problem 2: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{(-2)^k}{k^{10}}$ converges or diverges, using the Root Test.
Solution.

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{|a_k|} &= \lim_{k\to\infty} \sqrt[k]{\left|\frac{(-2)^k}{k^{10}}\right|} = \lim_{k\to\infty} \sqrt[k]{\frac{2^k}{k^{10}}} \\
&= \lim_{k\to\infty} \frac{2}{k^{10/k}} = 2 > 1.
\end{aligned}$$

$\xrightarrow{\text{Root Test}}$

$\sum_{k=1}^{\infty} a_k$ diverges.

Note: $\lim_{k\to\infty} k^{10/k} = 1$.
(Exercise: Verify $\lim_{x\to\infty} x^{10/x}$.)

Lesson 26

Topics: Choosing a Convergence Test
Misleading title !

Wrong Strategy:

Which test to choose ? Too many ! This is a wrong question to ask !

Right Strategy:

What does the given series look like ? This is the right question to ask !

$\longrightarrow$

∘ Find the close "model" series.

∘ Judge if the model series conveges or diverges, and guess by comparison or by analogy if the given series conveges or diverges.

∘ Justify your guess using a test.

Section Number: 10.8

Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 25. This should serve as a review for Lesson 25.
- Review for Lesson 25
- MyLabMath Homework for Lesson 25

(2) Example Problems

Example Problem 1: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}$ converges or diverges.

Solution.

∘ The series looks like $\sum_{k=1}^{\infty} \dfrac{2^k}{3^{k+1}}$, except for the annoying the second part $\sum_{k=1}^{\infty} \dfrac{\cos(\pi k)\sqrt{k}}{3^{k+1}}$

∘ What should we do with the second part $\sum_{k=1}^{\infty} \dfrac{\cos(\pi k)\sqrt{k}}{3^{k+1}}$?

Observe $\left\{ \begin{array}{lcl} |\cos(\pi k)| & \leq & 1, \\ \sqrt{k} & \leq & 2^k \text{(Show the graphs and compare.)} \end{array} \right.$

$\longrightarrow$

$\left|\cos(\pi k)\sqrt{k}\right| \leq 2^k$

Now

$$\begin{aligned} |a_k| &= \left|\frac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}\right| \\ &\leq \left|\frac{2^k}{3^{k+1}}\right| + \left|\frac{\cos(\pi k)\sqrt{k}}{3^{k+1}}\right| \\ &\leq \frac{2^k}{3^{k+1}} + \frac{2^k}{3^{k+1}} = 2 \cdot \frac{2^k}{3^{k+1}} := b_k \end{aligned}$$

circ $\sum b_k$ converges, since it is a geometric series with $r = \dfrac{2}{3} < 1$. By comparison, our guess is that $\sum a_k$ also converges.

∘ We only have to justify our guess.

$|a_k| \leq b_k$ and $\sum b_k$ converges.

$\xrightarrow{\text{C.T.}}$

$\sum |a_k|$ coverges.

$\xrightarrow{\text{A.C.T.}}$

$\sum a_k$ absolutely coverges.

Example Problem 2: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \dfrac{1}{\sqrt[4]{k^2-6k+9}}$ converges or diverges.

Solution.

$\circ$ Observe that $a_k = \dfrac{1}{\sqrt[4]{k^2-6k+9}}$ looks like $b_k = \dfrac{1}{\sqrt[4]{k^2}} = \dfrac{1}{k^{1/2}}$ when k is large, as $-6k+9$ is small compared to k^2.

$\circ$ Observe that $\sum b_k = \sum \dfrac{1}{k^{1/2}}$ diverges, since it is a p-series with $p = \dfrac{1}{2} \leq 1$. So our guess is that $\sum a_k$ should also diverge.

$\circ$ We only have to justify our guess.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{\sqrt[4]{k^2-6k+9}}}{\frac{1}{\sqrt[4]{k^2}}} = 1.$$

$\xrightarrow{\text{Limit Comparison Test}}$

$\sum a_k$ and $\sum b_k$ share the same destiny, and $\sum b_k$ diverges.

$\longrightarrow$

$\sum a_k$ diverges.

Example Problem 3: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = k^2 e^{-2k} = \dfrac{k^2}{(e^2)^k}$ converges or diverges.

Solution.

$\circ$ We compute

$$\begin{aligned} a_{k+1} &= \frac{(k+1)^2}{(e^2)^{k+1}} = \frac{(k+1)^2}{k^2} \cdot \frac{k^2}{(e^2)^k} \cdot \frac{1}{e^2} \\ &= \frac{(k+1)^2}{k^2} a_k \cdot \frac{1}{e^2}. \end{aligned}$$

When k is large, we have $\dfrac{(k+1)^2}{k^2} \sim 1$ (where "$\sim$" means "almost equal"). This implies, when k is large, $a_{k+1} \sim a_k \cdot \dfrac{1}{e^2}$. Therefore, the series looks like the geometric series $\sum b_k$ with $r = \dfrac{1}{e^2}$.

$\circ$ The series $\sum b_k$converges, since it is a geometric series with $r = \dfrac{1}{e^2} < 1$. So our guess is that $\sum a_k$ should also converge.

$\circ$ We only have to justify our guess.

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{(k+1)^2}{(e^2)^{k+1}}}{\frac{k^2}{(e^2)^k}} = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} \cdot \frac{1}{e^2} = \frac{1}{e^2} < 1$$

$\xrightarrow{\text{Ratio Test}}$

$\sum a_k$ absolutely converges.

Example Problem 4: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \sqrt[3]{\dfrac{k^2-1}{k^8+4}}$ converges or diverges.

Solution.

$\circ$ Observe that $a_k = \sqrt[3]{\dfrac{k^2-1}{k^8+4}}$ looks like $b_k = \sqrt[3]{\dfrac{k^2}{k^8}} = \dfrac{1}{k^2}$ when k is large, as -1 (resp. $+4$) is small compared to k^2 (resp. k^8).

$\circ$ Observe that $\sum b_k = \sum \dfrac{1}{k^2}$ diverges, since it is a p-series with $p = 2 > 1$. So our guess is that $\sum a_k$ should also converge.

$\circ$ We only have to justify our guess.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{\dfrac{k^2-1}{k^8+4}}}{\sqrt[3]{\dfrac{k^2}{k^8}}} = 1.$$

$\xrightarrow{\text{Limit Comparison Test}}$

$\sum a_k$ and $\sum b_k$ share the same destiny, and $\sum b_k$ converges.

$\longrightarrow$

$\sum a_k$ converges.

Example Problem 5: Determine whether the series $\sum_{k=1}^{\infty} a_k$ with $a_k = \left(1 - \dfrac{1}{10}\right)^k$ converges or diverges.

Solution.

$\circ$ What does the term $a_k = \left(1 - \dfrac{1}{10}\right)^k$ of the sequence look like when k is large ?

Since $\left(1 - \dfrac{1}{10}\right) \sim 1$ when k is large, is it true $a_k \sim 1$?

But since $\left(1 - \dfrac{1}{10}\right) < 1$, when k is large, $\left(1 - \dfrac{1}{10}\right)^k$ could become much smaller than 1 ?

$a_k = \left(1 - \dfrac{1}{10}\right)^k \sim ???$

We compute the limit $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1 - \dfrac{1}{10}\right)^k = e^{-10}$. (See Lesson 25 (3).) So the term a_k looks like $b_k = e^{-10}$.

$\circ$ The series $\sum b_k = \sum e^{-10}$ of coures diverges. So our guess is that $\sum a_k$ also diverges.

$\circ$ We only have to justify our guess.

$\lim_{k\to\infty} a_k = e^{-10} \neq 0$.

$\xrightarrow{\text{Divergence Test}}$

$\sum a_k$ diverges.

Lesson 27

Topics: Taylor Series Part 1
Subtitle: Approximate functions with polynomials
Section Number: 11.3, 11.1
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 26. This should serve as a review for Lesson 26.
- Review for Lesson 26
- MyLabMath Homework for Lesson 26

(2) What is a Taylor series of a function $f(x)$ centered at 0 ?
- Terminology:
  a power series expression of a function = a Taylor series of a function
- Discussion

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_kx^k + \cdots$$

Question: How can we find the coefficients $c_0, c_1, c_2, c_3, \ldots, c_k, \ldots$?
Answer:

$$f(0) = c_0 \longrightarrow c_0 = f(0).$$

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + kc_kx^{k-1} + \cdots$$

$$f'(0) = c_1 \longrightarrow c_1 = f'(0).$$

$$f''(x) = 2c_2 + 3 \cdot 2c_3x + \cdots + k(k-1)c_kx^{k-2} + \cdots$$

$$f''(0) = 2c_2 \longrightarrow c_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 3 \cdot 2c_3x + \cdots + k(k-1)(k-2)c_kx^{k-3} + \cdots$$

$$f'''(0) = 3 \cdot 2c_3 \longrightarrow c_3 = \frac{f'''(0)}{3 \cdot 2}$$

$\cdots$

$\cdots$

$$f^{(k)}(x) = k(k-1)(k-2)\cdots 2c_k + \cdots$$

$$f^{(k)}(0) = k(k-1)(k-2)\cdots 2c_k \longrightarrow c_k = \frac{f^{(k)}(0)}{k!}$$

- Definition:
  - **Taylor series centered at** 0 **(Maclaurin series)**

$$f(x) = \sum_{k=0}^{\infty} c_kx^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

  - **Taylor series centered at** a

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Examples

① $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$
$= \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}x^{2k+1}$

② $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$
$= \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k}$

③ $\cos x = 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$
$= \sum_{k=0}^{\infty} \frac{1}{k!}x^k$

④ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
$= \sum_{k=0}^{\infty} x^k$

(3) Taylor polynomials

- Definition:
  - ◦ **n-th order Taylor polynomial centered at** 0

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

- Idea behind it:

Taylor polynomial $P_n(x)$ approximates the function $f(x)$!
The bigger n is, the better the approximation is.

$$
\begin{array}{ccccc}
f(x) \;= & & \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k & & \\
& \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k & + & \sum_{k=n+1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \\
& \| & & \| \\
& P_n(x) & + & R_n(x) \\
& \| & & \| \\
& \text{n} - \text{th order T.P.} & & \text{Remainder}
\end{array}
$$

where the difference between $f(x)$ and $P_n(x)$, i.e., $|f(x) - P_n(x)| = |R_n(x)|$ is small when $|x|$ is small, since we have the estimate

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1}$$

where

$$|f^{(n+1)}(c)| \le M \text{ for any } c \in [0, x] \quad (\text{or } c \in [x, 0]).$$

- ◦ **n-th order Taylor polynomial centered at** a

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor polynomial $P_n(x)$ approximates the function $f(x)$!

$$\begin{array}{ccccc} f(x) = & & \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k & & \\ & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k & + & \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ & \| & & \| \\ & P_n(x) & + & R_n(x) \\ & \| & & \| \\ & \text{n} - \text{th order T.P.} & & \text{Remainder} \end{array}$$

where the difference between $f(x)$ and $P_n(x)$, i.e., $|f(x) - P_n(x)| = |R_n(x)|$ is small when $|x-a|$ is small, since we have the estimate

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

where

$$|f^{(n+1)}(c)| \le M \text{ for any } c \in [a,x] \quad (\text{or } c \in [x,a]).$$

Note: **1st order Taylor polynomial ceneterd at a is nothing but the linear approximation !**

$$\begin{aligned} P_1(x) &= \sum_{k=0}^{1} \sum_{k=0}^{1} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 \\ &= f(a) + f'(a)(x-a) = L(x). \end{aligned}$$

• Examples

①

$$\begin{cases} f(x) &= \ln x \\ f'(x) &= \dfrac{1}{x} \\ f''(x) &= -\dfrac{1}{x^2} \end{cases}$$

1st order Taylor polynomial for $f(x) = \ln x$ centered at 1

$$\begin{aligned} P_1(X) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 \\ &= f(1) + f'(1)(x-1) \\ &= 0 + \frac{1}{1}(x-1) = x-1. \end{aligned}$$

2nd order Taylor polynomial for $f(x) = \ln x$ centered at 1

$$\begin{aligned} P_2(X) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2 \\ &= f(1) + f'(1)(x-1) + \frac{1}{2}(x-1)^2 \\ &= 0 + \frac{1}{1}(x-1) + \frac{-\frac{1}{1^2}}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2. \end{aligned}$$

②

$$\left\{\begin{array}{rcl} f(x) & = & \sin x \\ f'(x) & = & \cos x \\ f''(x) & = & -\sin x \\ f'''(x) & = & -\cos x \end{array}\right.$$

$$\left\{\begin{array}{rcl} f^{(4)}(x) & = & \sin x \\ & & \text{(and the pattern repeats itself} \end{array}\right.$$

$$\left\{\begin{array}{rcl} f(0) & = & 0 \\ f'(0) & = & 1 \\ f''(0) & = & 0 \\ f'''(0) & = & -1 \end{array}\right.$$

$$\left\{\begin{array}{rcl} f^{(4)}(x) & = & 0 \\ & & \text{(and the pattern repeats itself} \end{array}\right.$$

Taylor polynomials for $f(x) = \sin x$ centered at $a = 0$.

$$\begin{array}{ccl} P_1(x) & = & x \\ \| & & \\ P_2(x) & & \\ P_3(x) & = & x - \dfrac{1}{3!}x^3 \\ \| & & \\ P_4(x) & & \\ P_5(x) & = & x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 \\ \| & & \\ P_6(x) & & \end{array}$$

Lesson 28

Topics: Taylor Series Part 2
Subtitle: Approximate functions with polynomials
Section Number: 11.3, 11.1
Lecture Plan:

(1) Use the first 15 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 27. This should serve as a review for Lesson 27.
- Review for Lesson 27
- MyLabMath Homework for Lesson 27

(2) Approximating the value (of a function) using the Taylor polynomial
- Example Problems

Example Problem 1:
(i) Approximate the value of $\sqrt{18}$ using the 3rd order Taylor polynomial for $f(x) = \sqrt{(x)}$ centered at $a = 16$.
(ii) Estimate the error $\left|\sqrt{18} - P_3(18)\right|$.

Solution.
We compute

$$\begin{cases} f(x) &= \sqrt{x} = x^{1/2} \\ f'(x) &= \frac{1}{2}x^{-1/2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} \\ f'''(x) &= \frac{3}{8}x^{-5/2} \\ f^{(4)}(x) &= -\frac{15}{16}x^{-7/2} \end{cases}$$

and hence

$$\begin{cases} f(16) &= 4 \\ f'(16) &= \frac{1}{4} \\ f''(16) &= -\frac{1}{256} \\ f'''(16) &= \frac{3}{8152} \\ f^{(4)}(16) &= -\frac{15}{16}\cdot\frac{1}{4^7} \end{cases}$$

Therefore, we have

$$\begin{aligned} P_3(x) &= \frac{f^{(0)}(16)}{0!}(x-16)^0 + \frac{f^{(1)}(16)}{1!}(x-16)^1 + \frac{f^{(2)}(16)}{2!}(x-16)^2 + \frac{f^{(3)}(16)}{3!}(x-16)^3 \\ &= 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2 + \frac{1}{16384}(x-16)^3 \end{aligned}$$

and hence

$$P_3(18) = 4 + \frac{1}{4} - \frac{1}{128} + \frac{1}{2048} = 4.24267578125$$

(ii) We have (See Lesson 27 (3): n-th order Taylor polynomial centered at a)

$$\left|\sqrt{18} - P_3(18)\right| = |f(18) - P_3(18)| \leq \frac{M}{(3+1)!}\,|18-16|^{3+1},$$

where we can set

$$M = \frac{15}{16}\cdot\frac{1}{4^7},$$

since we have for $c \in [16, 18]$ we have

$$\left|f^{(4)}(c)\right| = \left|-\frac{15}{16}\cdot\frac{1}{c^{7/2}}\right| \leq \frac{15}{16}\cdot\frac{1}{16^{7/2}} = \frac{15}{16}\cdot\frac{1}{4^7} = M.$$

We finally conclude

$$\left|\sqrt{18} - P_3(18)\right| \leq \frac{M}{(3+1)!}\,|18-16|^{3+1} = \frac{\frac{15}{16}\cdot\frac{1}{4^7}}{4!}\cdot 2^4 = \frac{5}{2^{17}}.$$

Example Problem 2:

(i) Approximate the value of $\ln\left(\frac{1}{2}\right)$ using the 3rd order Taylor polynomial for $f(x) = \ln(1-x)$ centered at $a = 0$.

(ii) Estimate the error $\left|\ln\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right)\right|$.

Solution.
We compute

$$\begin{cases} f(x) &= \ln(1-x) \\ f'(x) &= -(1-x)^{-1} \\ f''(x) &= -1\cdot(1-x)^{-2} \\ f'''(x) &= -1\cdot 2\cdot(1-x)^{-3} \\ f^{(4)}(x) &= -1\cdot 2\cdot 3\cdot(1-x)^{-4} \end{cases}$$

and hence

$$\begin{cases} f(0) &= 0 \\ f'(0) &= -1 \\ f''(0) &= -1 \\ f'''(0) &= -2 \\ f^{(4)}(0) &= -6 \end{cases}$$

Therefore, we have

$$\begin{aligned} P_3(x) &= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &= 0 - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \end{aligned}$$

and hence

$$\begin{aligned} P_3\left(\frac{1}{2}\right) &= 0 - \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\right)^3 \\ &= -\tfrac{2}{3} = -0.66666\cdots. \end{aligned}$$

(ii) We have (See Lesson 27 (3): n-th order Taylor polynomial centered at 0)

$$\left|\ln\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right)\right| = \left|f\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right)\right| \leq \frac{M}{(3+1)!}\left|\frac{1}{2}\right|^{3+1},$$

where we can set

$$M = 6\cdot 2^4,$$

since we have for $c \in [0, \frac{1}{2}]$ we have

$$\left|f^{(4)}(c)\right| = \left|-6(1-c)^{-4}\right| \leq 6\left(1-\frac{1}{2}\right)^{-4} = 6\cdot 2^4 = M.$$

We finally conclude

$$\left|\sqrt{18} - P_3(18)\right| \leq \frac{M}{(3+1)!}\left|\frac{1}{2}\right|^{3+1} = \frac{6\cdot 2^4}{4!}\left(\frac{1}{2}\right)^4 = \frac{1}{4}.$$

$\boxed{\text{Lesson 29}}$

Topics: Properties of Power Series Part 1
∘ radius of convergence
∘ interval of convergence
Section Number: 11.2
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 28. This should serve as a review for Lesson 28.
- Review for Lesson 28
- MyLabMath Homework for Lesson 28

(2) Main Question: Given a power series centered at a

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

for what value of x does the power series converge (or diverge) ?

(3) Example Problems

Example Problem 1: Given a power series centered at $a = 2$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(x-2)^k,$$

for what value of x does the power series converge (or diverge) ?

Solution. Set $a_k = \dfrac{(-1)^k}{4^k}(x-2)^k$.
Step 1. Ratio Test
We compute

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+1}}{4^{k+1}}(x-2)^{k+1}\right|}{\left|\dfrac{(-1)^k}{4^k}(x-2)^k\right|} \\ &= \frac{|x-2|}{4} \begin{cases} < 1 & \sum a_k \text{ converges} \\ = 1 & \text{inconclusive} \\ > 1 & \sum a_k \text{ diverges} \end{cases} \end{aligned}$$

$\longrightarrow$

$$|x-2| \begin{cases} < 4 & \sum a_k \text{ converges} \\ = 4 & \text{inconclusive} \\ > 4 & \sum a_k \text{ diverges} \end{cases}$$

$\longrightarrow$

R: the radius of convergence $= 4$.
Step 2. Check at the end points.

$\boxed{x = 6}$

$$\textstyle\sum_{k=0}^{\infty} \dfrac{(-1)^k}{4^k}(x-2)^k = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{\cancel{4^k}}\cancel{(6-2)^k}$$

diverges.

$\boxed{x = -2}$

$$\textstyle\sum_{k=0}^{\infty} \dfrac{(-1)^k}{4^k}(x-2)^k = \sum_{k=0}^{\infty} \dfrac{\cancel{(-1)^k}}{\cancel{4^k}}\cancel{(-2-2)^k}$$

diverges.
Step 3. Conclusion
∘ Explanation using a picture.
I: the interval of convergence $= (-2, 6)$.

Example Problem 2: Given a power series centered at $a = 0$

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k,$$

for what value of x does the power series converge (or diverge) ?

Solution. Set $a_k = \dfrac{1}{k!} x^k$.
Step 1. Ratio Test
We compute

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left| \frac{1}{(k+1)!} x^{k+1} \right|}{\left| \frac{1}{k!} x^k \right|} \\ &= \lim_{k\to\infty} \left| \frac{1}{k+1} x \right| = 0 < 1 \qquad (\text{fixing } x) \end{aligned}$$

This implies that, no matter what the value of x is, the power series $\sum_{k=0}^{\infty} \dfrac{1}{k!} x^k$ converges.
Step 2. N/A
Step 3. Conclusion
∘ Explanation using a picture.
I: the interval of convergence $= (-\infty, \infty)$.

Example Problem 3: Given a power series centered at $a = 0$

$$\sum_{k=0}^{\infty} k! x^k,$$

for what value of x does the power series converge (or diverge) ?

Solution. Set $a_k = k! x^k$.
Step 1. Ratio Test
We compute

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|(k+1)! x^{k+1}\right|}{|k! x^k|} \\ &= \lim_{k\to\infty} |(k+1)x| = \infty < 1 \qquad (\text{fixing } x) \end{aligned}$$

This implies that, no matter what the value of x is, the power series $\sum_{k=0}^{\infty} k! x^k$ diverges.
Step 2. N/A
Step 3. Conclusion
∘ Explanation using a picture.
I: the interval of convergence $= [0, 0]$.

Example Problem 4: Given a power series centered at $a = 2$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} (x-2)^k,$$

for what value of x does the power series converge (or diverge) ?

Solution. Set $a_k = \dfrac{1}{\sqrt{k}}(x-2)^k$.

Step 1. Ratio Test

We compute

$$\begin{aligned}
\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|\frac{1}{\sqrt{k+1}}(x-2)^{k+1}\right|}{\left|\frac{1}{\sqrt{k}}(x-2)^k\right|} \\
&= \lim_{k\to\infty} \left|\frac{\sqrt{k}}{\sqrt{k+1}}(x-2)\right| \\
&= |x-2| \quad \begin{cases} <1 & \sum a_k \text{ converges} \\ =1 & \text{inconclusive} \\ >1 & \sum a_k \text{ diverges} \end{cases}
\end{aligned}$$

○ Explanation using a picture.

R: the radius of convergence $= 1$.

Step 2. Check at the end points.

$\boxed{x=3}$

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k}}(x-2)^k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}}(3-2)^k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=0}^{\infty} \frac{1}{k^{1/2}}$$

diverges, since it is a p-series with $p = \dfrac{1}{2} \leq 1$.

$\boxed{x=1}$

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k}}(x-2)^k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}}(1-2)^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$$

converges by the Alternating Series Test.

Step 3. Conclusion

○ Explanation using a picture.

I: the interval of convergence $= [1, 3)$.

Lesson 30

Topics: Summary of Taylor Series and Applications
∘ How to find the Taylor Series of a function easily
∘ Computing the limits using the Taylor Series
Section Number: 11.3, 11.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 29. This should serve as a review for Lesson 29.
- Review for Lesson 29
- MyLabMath Homework for Lesson 29

(2) Summary
- Taylor series
  - ∘ Taylor series centered at a

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

  - ∘ Taylor series centered at $a=0$ (has the name Maclaurin series)

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

- Examples

$$\begin{array}{rcl} ① \sin x & = & \sum_{k=0}^{\infty}(-1)^k\dfrac{1}{(2k+1)!}x^{2k+1} \\ & = & x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5-\dfrac{1}{7!}x^7+\cdots \\ ② \cos x & = & \sum_{k=0}^{\infty}(-1)^k\dfrac{1}{(2k)!}x^{2k} \\ & = & 1-\dfrac{1}{2!}x^2+\dfrac{1}{4!}x^4-\dfrac{1}{6!}x^6+\cdots \\ ② e^x & = & \sum_{k=0}^{\infty}\dfrac{1}{k!}x^k \\ & = & 1+\dfrac{1}{1!}x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+\cdots \end{array}$$

(3) Example Problems

Example Problem 1: Find the Taylor series (power series expression) centered at $a=0$ for

$$f(x)=\frac{1}{1-x}.$$

Find also its radius of convergence and the interval of convergence.

Solution.
We compute

$$\left\{\begin{array}{rcl} f(x) & = & \dfrac{1}{1-x}=(1-x)^{-1} \\ f'(x) & = & 1\cdot(1-x)^{-2} \\ f''(x) & = & 1\cdot 2\cdot(1-x)^{-3} \\ & & \cdots \\ f^{(k)}(x) & = & 1\cdot 2\cdot 3\cdot\cdots\cdot k(1-x)^{-(k+1)} \end{array}\right.$$

and hence

$$\left\{\begin{array}{rcl} f(0) & = & 1 \\ f'(0) & = & 1! \\ f''(0) & = & 2! \\ & & \cdots \\ f^{(k)}(0) & = & k! \end{array}\right.$$

Therefore, we have

$$\begin{aligned} \frac{1}{1-x} &= \textstyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= \textstyle\sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

Note (Easier Solution): Just carry out the long division !

What is R ? What is I ?

Set $a_k = x^k$.
Step 1. Ratio Test
We compute

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{|x^{k+1}|}{|x^k|} \\ &= |x| \begin{cases} < 1 & \sum a_k \text{ converges} \\ = 1 & \text{inconclusive} \\ > 1 & \sum a_k \text{ diverges} \end{cases} \end{aligned}$$

∘ Explanation using a picture.
R: the radius of convergence $= 1$.
Step 2. Check at the end points.
$\boxed{x = 1}$

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} 1^k$$

diverges.
$\boxed{x = -1}$

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} (-1)^k$$

diverges.
Step 3. Conclusion
∘ Explanation using a picture.
I: the interval of convergence $= (-1, 1)$.
Note: The function is defined by the formula $f(x) = \dfrac{1}{1-x}$ and hence $f(-1) = \dfrac{1}{1-(-1)} = \dfrac{1}{2}$ is well-defined, while the power series $\sum_{k=0}^{\infty} x^k$ is divergent at $x = -1$.
Example Problem 2: Find the Taylor series (power series expression) centered at $a = 0$ for

$$f(x) = \frac{x^5}{1-x}.$$

Find also its radius of convergence and the interval of convergence.

Solution.
We have
$\dfrac{x^5}{1-x} = x^5 \cdot \left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=5}^{\infty} x^k.$
We also have
$\{x^5 \cdot \left(\sum_{k=0}^{\infty} x^k\right) \text{ converges}\} \Longleftrightarrow \{\sum_{k=0}^{\infty} x^k \text{ converges}\} \Longleftrightarrow \{x \in (-1, 1)\}$
$\longrightarrow$
$R = 1$ & $I = (-1, 1)$.

Example Problem 3: Find the Taylor series (power series expression) centered at $a = 0$ for

$$f(x) = \frac{1}{1-2x}.$$

Find also its radius of convergence and the interval of convergence.

Solution.
Set $2x = X$.
Then we have

$$\begin{aligned} \frac{1}{1-2x} &= \frac{1}{1-X} = \sum_{k=0}^{\infty} X^k \quad \text{(See Example Problem 1)} \\ &= \sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} 2^k x^k \end{aligned}$$

Moreover, we have

$\{\sum_{k=0}^{\infty} X^k \text{ converges}\} \Longleftrightarrow \{X \in (-1,1)\}$

$\longrightarrow$

$\{\sum_{k=0}^{\infty} (2x)^k \text{ converges}\} \Longleftrightarrow \{2x \in (-1,1)\} \Longleftrightarrow \{x \in \left(-\frac{1}{2}, \frac{1}{2}\right)\}$

$\longrightarrow$

$R = \frac{1}{2}$ & $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Example Problem 4: Find the Taylor series (power series expression) centered at $a = 0$ for

$$f(x) = \frac{1}{1+x^2}.$$

Find also its radius of convergence and the interval of convergence.

Solution.
Set $-x^2 = X$.
Then we have

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \frac{1}{1-X} = \sum_{k=0}^{\infty} X^k \quad \text{(See Example Problem 1)} \\ &= \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} \end{aligned}$$

Moreover, we have

$\{\sum_{k=0}^{\infty} X^k \text{ converges}\} \Longleftrightarrow \{X \in (-1,1)\}$

$\longrightarrow$

$\{\sum_{k=0}^{\infty} (-x^2)^k \text{ converges}\} \Longleftrightarrow \{-x^2 \in (-1,1)\} \Longleftrightarrow \{x \in (-1,1)\}$

$\longrightarrow$

$R = 1$ & $I = (-1,1)$.

(4) Some Applications

- Computation of limits

○ Example 1: Compute

$$\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{3x^4}.$$

Solution.

Since it is formally of the form $\left(\frac{0}{0}\right)$, we could compute the limit using L'Hospital's Rule.

Here we compute the limit using the Taylor series of the functions as follows.
Since

$$\begin{aligned} \cos x &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + (\text{higher terms than } x^4), \end{aligned}$$

we have

$$\lim_{x\to 0}\frac{x^2+2\cos x-2}{3x^4}$$

$$=\lim_{x\to 0}\frac{\cancel{x^2}+2\left\{\cancel{1}-\cancel{\frac{1}{2}x^2}+\frac{1}{24}x^4+(\text{higher terms than } x^4)\right\}-\cancel{2}}{3x^4}$$

$$=\lim_{x\to 0}\frac{\frac{1}{12}x^4+(\text{higher terms than } x^4)}{3x^4}$$

$$=\lim_{x\to 0}\left[\frac{1}{36}+(\text{higher terms than } x^0)\right]=\frac{1}{36}.$$

○ Example 2: Compute

$$\begin{array}{cccccccc} \lim_{x\to\infty} & \left[6x^5\right. & \cdot & \sin\left(\dfrac{1}{x}\right) & - & 6x^4 & + & \left.x^2\right] \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \infty & \times & 0 & - & \infty & + & \infty \end{array} \quad (???)$$

It is hard to compute the given limit as it is.

Set $t=\frac{1}{x}$ $(\longrightarrow x=\frac{1}{t})$.

We have $t\to t^+$ as $x\to\infty$.

Now we compute

$$\begin{aligned}\lim_{x\to\infty}\left[6x^5\cdot\sin\left(\frac{1}{x}\right)-6x^4+x^2\right] &= \lim_{t\to 0^+}\left[6\cdot\frac{1}{t^5}\cdot\sin t-6\cdot\frac{1}{t^4}+\frac{1}{t^2}\right]\\ &= \lim_{t\to 0^+}\frac{6\sin t-6t+t^3}{t^5}.\end{aligned}$$

Since the last limit is formally of the form $\left(\frac{0}{0}\right)$, we could compute the limit using L'Hospital's Rule.

Here we compute the limit using the Taylor series of the functions as follows.

Since

$$\begin{aligned}\sin t &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)!}t^{2k+1}\\ &= t-\frac{1}{6}t^3+\frac{1}{120}t^5+(\text{higher terms than } t^5),\end{aligned}$$

we have

$$\lim_{t\to 0^+}\frac{6\sin t-6t+t^3}{t^5}$$

$$=\lim_{t\to 0^+}\frac{6\left\{\cancel{t}-\cancel{\frac{1}{6}t^3}+\frac{1}{120}t^5+(\text{higher terms than } t^5)\right\}-\cancel{6t}+\cancel{t^3}}{t^5}$$

$$=\lim_{t\to 0^+}\frac{\frac{1}{20}t^5+(\text{higher terms than } t^5)}{t^5}$$

$$=\lim_{x\to 0}\left[\frac{1}{20}+(\text{higher terms than } t^0)\right]=\frac{1}{20}.$$

$\boxed{\text{Lesson 31}}$

Topics: Properties of Power Series Part 2
Section Number: 11.2, 11.4
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 30. This should serve as a review for Lesson 30.
- Review for Lesson 30
- MyLabMath Homework for Lesson 30

(2) Differentiating and Integrating Power Series
- Principle: Term by term
- Summary

$$f(x) = \sum c_k (x-a)^k$$

◦ Differentiation

$$f'(x) = \sum c_k \cdot k(x-a)^{k-1}$$

$$\begin{cases} R \text{ remains the same} \\ I \text{ may change} \end{cases}$$

◦ Integration

$$\int f(x)\ dx = \sum c_k \cdot \frac{1}{k+1}(x-a)^{k+1} + C$$

$$\begin{cases} R \text{ remains the same} \\ I \text{ may change} \end{cases}$$

- Examples

We know (See Exampole Problem 1 of (3) in Lesson 30.)

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

with $\begin{cases} R = 1 & \text{: radius of convergence} \\ I = (-1, 1) & \text{: interval of convergence.} \end{cases}$

◦ Differentiating, we have

$$f'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} kx^{k-1}.$$

The radius of convergence remains the same, i.e., $R = 1$. This implies that the center is 0, and the end points are ± 1. After differentiating, we may observe that the interval of convergence changes. Therefore, in order to determine the interval of convergence, we have to check the behavior at the end points.

$\boxed{x = 1}$

$$\sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} k \cdot 1^{k-1}$$

diverges.

$\boxed{x = -1}$

$$\sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} k \cdot (-1)^{k-1}$$

diverges.

Therefore, we conclude
$$I = (-1, 1).$$

◦ Integrating, we have
$$\int f(x)\ dx = -\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + C = \sum_{k=0}^{\infty} \frac{1}{k+1}x^{k+1} + C.$$
The radius of convergence remains the same, i.e., $R = 1$. This implies that the center is 0, and the end points are ± 1. After integrating, we may observe that the interval of convergence changes. Therefore, in order to determine the interval of convergence, we have to check the behavior at the end points.

$\boxed{x = 1}$
$$\sum_{k=0}^{\infty} \frac{1}{k+1}x^{k+1} + C = \sum_{k=0}^{\infty} \frac{1}{k+1}1^{k+1} + C = \sum_{k=0}^{\infty} \frac{1}{k+1} + C$$
diverges, since it is the harmonic series.

$\boxed{x = -1}$
$$\sum_{k=0}^{\infty} \frac{1}{k+1}x^{k+1} + C = \sum_{k=0}^{\infty} \frac{1}{k+1}(-1)^{k+1} + C = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{k+1} + C$$
converges by the Alternating Seroes Test.

Therefore, we conclude
$$I = [-1, 1).$$

(3) Example Problems

Example Problem 1: Find the power series expression for
$$f(x) = \tan^{-1}(x)$$
and its radius of convergence and the interval of convergence.

Solution.
Set $f(x) = \tan^{-1}(x)$.
Then
$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} \\ &= \textstyle\sum_{k=0}^{\infty} x^{2k} \quad \text{(See Example Problem 4 in Lesson 30 (3)} \\ &= 1 - x^2 + x^4 - x^6 + \cdots \end{aligned}$$
The power series has
$$\begin{cases} R = 1 & : \text{radius of convergence} \\ I = (-1, 1) & : \text{interval of convergence.} \end{cases}$$
Therefore, by the principle and summary above, we have
$$\begin{aligned} f(x) &= \int f'(x)\ dx \\ &= \textstyle\sum_{k=0}^{\infty}(-1)^k \dfrac{1}{2k+1}x^{2k+1} + C. \end{aligned}$$
In order to determine what the integration constant C is, we plug in $x = 0$. Then we have
$$\begin{aligned} f(0) &= \tan^{-1}(0) = 0 \\ &= \textstyle\sum_{k=0}^{\infty}(-1)^k \dfrac{1}{2k+1}0^{2k+1} + C \end{aligned} \quad \longrightarrow C = 0$$
$\longrightarrow$
$$f(x) = \tan^{-1} x = \textstyle\sum_{k=0}^{\infty}(-1)^k \dfrac{1}{2k+1}x^{2k+1}.$$
By the principle and summary above again, we have
$$\begin{cases} R = 1 & : \text{radius of convergence} \\ I = ??? & : \text{interval of convergence.} \end{cases}$$

We have to analyze the behavior of the power series at the end points to determine the interval of convergence I.

$\boxed{x = 1}$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}x^{2k+1} = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}$$

converges by the Alternating Series Test.

$\boxed{x = -1}$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}x^{2k+1} = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}(-1)^{2k+1} = \sum_{k=0}^{\infty}(-1)^{k+1} \frac{1}{2k+1}$$

converges by the Alternating Series Test.

$\longrightarrow$

$$I = [-1, 1].$$

Example Problem 2: Find the power series expression for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ and its radius of convergence and the interval of convergence.

Solution.
Observe

$$\begin{cases} \ln(1-x) &= -\sum_{k=1}^{\infty} \frac{1}{k}x^k \\ \ln(1+x) &= \ln(1-(1-x)) \\ &= -\sum_{k=1}^{\infty} \frac{1}{k}(-x)^k \\ &= -\sum_{k=1}^{\infty} \frac{1}{k}(-1)^k x^k \end{cases}$$

Therefore, we compute

$$\begin{aligned} f(x) &= \ln\left(\frac{1+x}{1-x}\right) \\ &= \ln(1+x) - \ln(1-x) \\ &= \left\{-\sum_{k=1}^{\infty} \frac{1}{k}(-1)^k x^k\right\} - \left\{-\sum_{k=1}^{\infty} \frac{1}{k}x^k\right\} \\ &= \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right\} - \left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdot\right\} \\ &= 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right\} \\ &= 2\sum_{k=0}^{\infty} \frac{1}{2k+1}x^{2k+1}. \end{aligned}$$

Set $a_k = \frac{1}{2k+1}x^{2k+1}$.

Step 1. Ratio Test

We compute

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|\frac{1}{2(k+1)+1}x^{2(k+1)+1}\right|}{\left|\frac{1}{2k+1}x^{2k+1}\right|} \\ &= \lim_{k\to\infty} \left|\frac{2k+1}{2k+3}x^2\right| \\ &= |x|^2 \begin{cases} <1 & \sum a_k \text{ converges} \\ =1 & \text{inconclusive} \\ >1 & \sum a_k \text{ diverges} \end{cases} \end{aligned}$$

$\longrightarrow$

$$|x| \begin{cases} <1 & \sum a_k \text{ converges} \\ =1 & \text{inconclusive} \\ >1 & \sum a_k \text{ diverges} \end{cases}$$

$\longrightarrow$

R: the radius of convergence $= 1$.

Step 2. Check at the end points.
$\boxed{x=1}$

$$\sum_{k=0}^{\infty}\frac{1}{2k+1}x^{2k+1}=\sum_{k=0}^{\infty}\frac{1}{2k+1}1^{2k+1}=\sum_{k=0}^{\infty}\frac{1}{2k+1}$$

diverges by the Limit Comparison Test with $b_k=\dfrac{1}{k}$.
$\boxed{x=1}$

$$\sum_{k=0}^{\infty}\frac{1}{2k+1}x^{2k+1}=\sum_{k=0}^{\infty}\frac{1}{2k+1}(-1)^{2k+1}=-\sum_{k=0}^{\infty}\frac{1}{2k+1}$$

diverges.
Step 3. Conclusion
○ Explanation using a picture.
 I: the interval of convergence $=(-1,1)$.

Example Problem 2: Verify

$$\frac{d}{dx}(\sin x)=\cos x$$

using the power series.

Solution.
Writing down the power series centered at $a=0$ (i.e., Maclaurin series) for $\sin x$, we have

$$\sin x=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}x^{2k+1}.$$

Differentiating, we obtain

$$\begin{aligned}(\sin x)' &= \left\{\textstyle\sum_{k=0}^{\infty}\dfrac{1}{(2k+1)!}x^{2k+1}\right\}\\ &= \textstyle\sum_{k=0}^{\infty}\dfrac{1}{(2k+1)!}(2k+1)x^{2k}\\ &= \textstyle\sum_{k=0}^{\infty}\dfrac{1}{(2k)!}x^{2k}=\cos x.\end{aligned}$$

Example Problem 3 (Representing numbers as infinite series (and vice versa):
Compute the exact value of the infinite series

$$1-\frac{1}{3}+\frac{1}{5}-\cdots=\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}.$$

Solution.
Step 1. Check the convergence of the series.
 Set $b_k=\dfrac{1}{2k+1}$.
Check
Condition ① b_k decreasing ?
 Yes, because $b_k=\dfrac{1}{2k+1}\geq\dfrac{1}{2(k+1)+1}=b_{k+1}$ ✓
Condition ② $\lim_{k\to\infty}b_k=0$?
 Yes, because $\lim_{k\to\infty}b_k=\lim_{k\to\infty}\dfrac{1}{2k+1}=0$ ✓
Therefore, by the A.S.T., we conclude that the series $\sum_{k=0}^{\infty}(-1)^kb_k=\sum_{k=0}^{\infty}(-1)^k\dfrac{1}{2k+1}$ converges to some finite number S.

Step 2. What is the Maclaurin series for $tan^{-1}x$?
We try to figure out the power series centered at $a = 0$ of $tan^{-1}x$ (even though this seems irrelevant to our main question "What is S ?").
For this purpose, we observe

$$\begin{aligned}(\tan^{-1} x)' &= \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \\ &= \textstyle\sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}.\end{aligned}$$

Integrating back, we obtain

$$\begin{aligned}\tan^{-1} x &= \int (\tan^{-1} x)' \, dx \\ &= \int \left\{ \sum_{k=0}^{\infty}(-1)^k x^{2k} \right\} \, dx \\ &= \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C.\end{aligned}$$

Finally, plugging in $x = 0$, we obtain $0 = \tan^{-1}(0) = C$ and hence

$$tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

Step 3. What is S ?
We plug in $x = 1$ to the abive power series expression for $\tan^{-1} x$ cneterd at $a = 0$ to obtain

$$\pi/4 = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}.$$

item Representing functions as power series or vice versa (Optional)
• Examples
○ Example 1: Consider the power series

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k!},$$

and we look for its expression as a function.
Reacll that

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

Setting $X = 1 - 2x$, we obtain the desired expression

$$e^{1-2x} = \sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}.$$

○ Example 2: Consider the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} k x^{2k} = \sum_{k=1}^{\infty} k \left(-\frac{x^2}{4} \right)^k$$

and we look for its expression as a function.
Observe

$$\frac{1}{1-X} = \sum_{k=0}^{\infty} X^k.$$

Differentiating, we obtain

$$\begin{array}{ccc} \left(\dfrac{1}{1-X}\right) & = & \sum_{k=0}^{\infty} kX^{k-1} \\ \| & & \| \\ \dfrac{1}{(1-X)^2} & & \sum_{k=1}^{\infty} kX^{k-1} \end{array}$$

and hence

$$\frac{X}{(1-X)^2} = X \cdot \sum_{k=1}^{\infty} kX^{k-1} = \sum_{k=1}^{\infty} kX^k.$$

Finally setting

$$X = -\frac{x^2}{4},$$

we obtain

$$\begin{array}{ccc} \dfrac{\left(-\dfrac{x^2}{4}\right)}{\left\{1-\left(-\dfrac{x^2}{4}\right)\right\}^2} & = & \sum_{k=1}^{\infty} k\left(-\dfrac{x^2}{4}\right)^k \\ \| & & \\ \dfrac{-4x^2}{(4+x^2)^2}. & & \end{array}$$

Lesson 32

Topics: Polar Coordinates (Basics)
Section Number: 12.2
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 31. This should serve as a review for Lesson 31.
- Review for Lesson 31
- MyLabMath Homework for Lesson 31

(2) Comparison between Cartesian coordinates and polar coordinates
- Explanation using a picture
- Basic relations
  - (x, y) Cartesian coordinates
  - (r, θ) Polar coordinates

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases} \quad \& \quad \begin{cases} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

Warning: Even if we fix a point, its polar coordnates are not uniquely determined, while its Cartesian coordinates are uniquely determined. The second rlation holds for one choice of its polar coordinates.
- Example: Explanation using a picture
  - $(x, y) = (2, 2)$ Cartesian coordinates
  - $(r, \theta) = (2\sqrt{2}, \pi/4 + 2\pi n)$ or $(-2\sqrt{2}, 5\pi/4 + 2\pi n)$ for $n \in \mathbb{Z}$
    Polar coordinates

Note: Emphasize that "r" can be negative.

(3) Equation of figures in Polar coordinates
- Examples (Explanation using a figure)
  1. $\theta = \dfrac{\pi}{3}$ (line)
  2. $r = \dfrac{5}{3\cos\theta + 4\sin\theta}$ (line)
  3. $r = 3$ (circle)
  4. $r = \sin(2\theta)$ (four-leafed clover)
  5. $r = 2\sin\theta$ (circle)
     - Geometric meaning as well as the algebraic manupulation

  Exercise: $r = 2\cos\theta$
  6. $r = 2\cdot 5\cos\theta + 2\cdot 12\sin\theta$ (circle)
  7. (cardioids)

$$\begin{cases} r &= f(\theta) = 1+\sin\theta \\ r &= f(\theta) = 1+0.7\sin\theta \\ r &= f(\theta) = 1+2\sin\theta \end{cases}$$

Lesson 33

Topics: Calculus in polar coordinates Part 1
Section Number: 12.3
Lecture Plan:

(1) Use the first 15 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 32. This should serve as a review for Lesson 32.
- Review for Lesson 32
- MyLabMath Homework for Lesson 32

(2) Slope of a tangent
- Formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\{r = f(\theta) \quad \text{emphasizing } r \text{ is a function of } \theta$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

- Example Problems

Example Problem 1: Consider the circle of radius 5 with center being the origin.

What is the slope of the tangent to the circle at point $\left(\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}\right)$?

Solution.
Draw a picture !
It is clear from the picture that

$$\frac{dy}{dx} = -1.$$

We confirm and check this fact using our formula above

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &\overset{r'=0}{=} -\frac{\cos\theta}{\sin\theta} \overset{\theta=\pi/4}{=} -1. \end{aligned}$$

Example Problem 2: Consider the cardioid given by the equation

$$r = 1 - \cos\theta \quad -\pi \le \theta \le \pi.$$

Find the points on the cardioid where the tangent is horizontal.

Note: Since the cardioid has a singularity at the origin, where the tangent is not well-defined, we exclude the corresponding angle $\theta = 0$ from the consideration. Also when $\theta = -\pi$ or π, we are at the point $(-1, 0)$ where the tangent line is vertical. We also exclude these angles from consideration.

Solution.
Draw a picture !
We compute

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\
&= = \frac{(1-\cos\theta)'\sin\theta + (1-\cos\theta)\cos\theta}{(1-\cos\theta)'\cos\theta - (1-\cos\theta)\sin\theta} \\
&= = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta} \\
&= \frac{(1-\cos^2\theta) + (1-\cos\theta)\cos\theta}{\sin\theta\{\cos\theta - (1-\cos\theta)\}} \\
&= \frac{-2\cos^2\theta + \cos\theta + 1}{\sin\theta\{2\cos\theta - 1\}} \\
&= -\frac{(2\cos\theta+1)(\cos\theta-1)}{\sin\theta\{2\cos\theta-1\}}
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
\text{tangent being horizontal} &\iff \frac{dy}{dx} = 0 \\
&\overset{\theta\neq\pm\pi,0}{\iff} 2\cos\theta + 1 = 0 \\
&\iff \cos\theta = -\frac{1}{2} \\
&\iff \theta = \pm\frac{2\pi}{3}.
\end{aligned}$$

Exercise: Show that

$$\text{tangent being vertical} \iff \theta = \pm\pi, \pm\frac{\pi}{3}$$

Lesson 34

Topics: Calculus in polar coordinates Part 2
Section Number: 12.3
Lecture Plan:

(1) Use the first 10 mimutes to discuss some difficult problems from MyLabMath HW for Lesson 33. This should serve as a review for Lesson 33.
- Review for Lesson 33
- MyLabMath Homework for Lesson 33

(2) Area of the region bounded by polar curves
- Explanation of the idea using a picture
- Formula

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$

with the last formula emphasizing that r is a function of θ)
- Example Problem: Find the area of the four-leafed rose defined by the polar equation

$$r = 3\cos(2\theta) \quad 0 \leq \theta \leq 2\pi.$$

Solution.
Step 1. Discuss how to draw the four-leafed rose.
Step 2. Compute the area of half of a petal.
Draw a picture.
We compute

$$\begin{aligned} \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta &= \int_0^{\pi/4} \frac{1}{2} \{3\cos(2\theta)\}^2 \, d\theta \\ &= \frac{9}{2} \int_0^{\pi/4} \cos^2(2\theta) \, d\theta \\ &\overset{\text{double angle formula}}{=} \frac{9}{2} \int_0^{\pi/4} \frac{1+\cos(4\theta)}{2} \, d\theta \\ &= \frac{9}{4} \left[\theta + \frac{1}{4}\sin(4\theta) \right]_0^{\pi/4} = \frac{9\pi}{16}. \end{aligned}$$

Step 3. Total area $= 8 \times$ (area of half of a petal) $= 8 \times \dfrac{9\pi}{16} = \dfrac{9\pi}{2}$.

(3) Arc Length
- Formula

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\ &= \int_a^b \sqrt{dx^2 + dy^2} \end{aligned}$$

$$\left\{ \begin{array}{lcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array} \right.$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$
\begin{aligned}
&= \int_\alpha^\beta \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\ d\theta \\
&\left\{\begin{array}{l}
(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2 \\
= (r')^2\cos^2\theta + r^2\sin^2\theta + (r')^2\sin^2\theta + r^2\cos^2\theta \\
\cancel{-2r'r\cos\theta\sin\theta} + \cancel{2r'r\sin\theta\cos\theta} \\
= (r')^2 + r^2
\end{array}\right. \\
&= \int_\alpha^\beta \sqrt{(r')^2 + r^2}\ d\theta
\end{aligned}
$$

• Example Problem: Compute the arc length of the cardioid defined by

$$r = 1\cos\theta \quad 0 \le \theta \le 2\pi.$$

Solution.
Draw the picture.
We compute

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(r')^2 + r^2}\ d\theta \\
&= \int_0^{2\pi} \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\ d\theta \quad \longleftarrow (r' = -\sin\theta) \\
&= \int_0^{2\pi} \sqrt{\sin^2\theta + (1 + 2\cos\theta + \cos^2\theta)}\ d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\ d\theta \\
&\left\{\begin{array}{l}
\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2} \\
\longrightarrow \\
4\cos^2\left(\frac{\theta}{2}\right) = 2 + 2\cos\theta
\end{array}\right. \\
&= \int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\ d\theta \\
&= \int_0^{2\pi} \left|2\cos\left(\frac{\theta}{2}\right)\right|\ d\theta
\end{aligned}
$$

Now we continue our computation paying attention to when $\cos\left(\frac{\theta}{2}\right)$ is positive or negative.

$$
\begin{aligned}
&= \int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right)\ d\theta + \int_\pi^{2\pi} \left\{-2\cos\left(\frac{\theta}{2}\right)\right\}\ d\theta \\
&= \left[4\sin\left(\frac{\theta}{2}\right)\right]_0^\pi + \left[-4\sin\left(\frac{\theta}{2}\right)\right]_\pi^{2\pi} \\
&= 4 + 4 = 8.
\end{aligned}
$$

Lesson 35

Topics: Summary of Polar Coordinates
Section Number: 12.2, 12.3
Lecture Plan:

The subjects of Polar Coordinates and Calculus in Polar Coordinates (especially the latter) are formidable both in quantity and difficulty for the students to digest. Most likely the instructor cannot cover everything scheduled to be covered in Lessons 32, 33, 34. This lesson is reserved as a shock absorber so that the instructor can catch up with the schedule and/or review the materials.