MA161 **FINAL EXAM** **December 1998**

Name: SOLUTION KEY

I.D.#: ____________

Recitation Instructor: ____________ **Time of Recitation** ________

Lecturer: ____________ **Section#:** ____________

Instructions:

1. Fill in your name, student ID number and division and section number on the mark-sense sheet. Also fill out the information requested above.
2. This booklet consists of 9 pages. There are 25 questions, each worth 8 points.
3. Mark your answers on the mark-sense sheet. Please show your working in this booklet.
4. No books, notes or calculators may be used.
5. When you are finished with the exam hand this booklet and the mark-sense sheet, in person, to your instructor.

1. The function $f(x) = x^3 - 9x + 1$ has two critical points. The line through these two points has slope

$f'(x) = 3x^2 - 9 = 0 \Rightarrow x = \pm\sqrt{3}$

$(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, 3\sqrt{3} - 9\sqrt{3} + 1) = (\sqrt{3}, -6\sqrt{3} + 1)$

$(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, -3\sqrt{3} + 9\sqrt{3} + 1) = (-\sqrt{3}, 6\sqrt{3} + 1)$

slope: $\dfrac{(-6\sqrt{3}+1) - (6\sqrt{3}+1)}{\sqrt{3} - (-\sqrt{3})} = \dfrac{-12\sqrt{3}}{2\sqrt{3}} = -6$

A. -6
B. $-\sqrt{3}$
C. 0
D. $\sqrt{3}$
E. 6

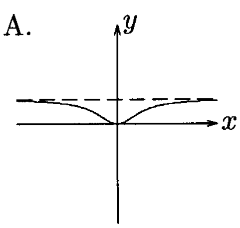
2. Evaluate $\lim_{x \to 0^+} x \csc^2 x = \lim_{x \to 0^+} \dfrac{x}{\sin^2 x}$

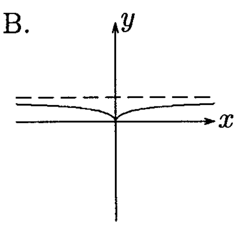
$= \lim_{x \to 0^+} \left(\dfrac{x}{\sin x} \cdot \dfrac{1}{\sin x} \right)$

$= 1 \cdot \dfrac{1}{0^+} = \infty$

A. $-\infty$
B. -1
C. 0
D. 1
E. ∞

3. Which could be the graph of $f(x) = \dfrac{x^3}{1 + |x|^3}$

A.


B.


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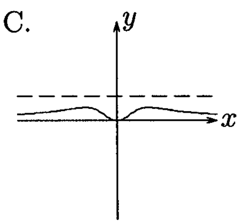
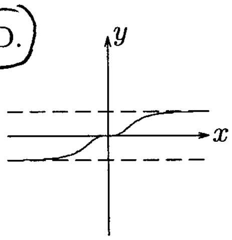
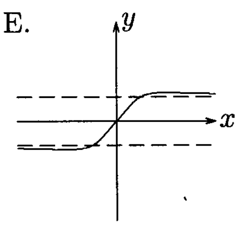


D.


E.


$f(x)$ has sign of x.
$\therefore$ A, B, C eliminated.

Note: if $x > 0$, then $0 < \dfrac{x^3}{1+|x|^3} < 1$
if $x < 0$, then $-1 < \dfrac{x^3}{1+|x|^3} < 0$
$\Rightarrow$ graph is D

4. What value of a makes the following function continuous at $x = 0$

$$f(x) = \begin{cases} 2\cos x & x < 0 \\ 3\sin x + a & x \geq 0 \end{cases}$$

Want: $f(0) = \lim_{x \to 0} f(x)$.

$f(0) = 3\sin 0 + a = 3(0) + a = a$

$\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} 2\cos x = 2$

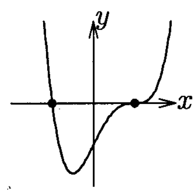
$\Rightarrow \lim_{x \to 0^+} f(x)$ must be 2 also.

$\Rightarrow a = 2$

A. -2
B. -1
C. 0
D. 1
E. 2

5. $f(x) = (x+1)^p(x-1)^q$. The graph of $f(x)$ is

Graph must be C.



Note: A $(1,1) \to$
$f(x) = (x+1)(x-1)$
$= x^2 - 1$
$\Rightarrow$ not A.

The pair of integers (p, q) could be

A.

		-1		1	
$x+1$	−		+		+
$x-1$	−		−		+
$f(x)$	+		−		+

C.

		-1		1	
$x+1$	−		+		+
$(x-1)^3$	−		−		+
$f(x)$	+		−		+

E.

		-1		1	
$(x+1)^2$	+		+		+
$(x-1)$	−		−		+
$f(x)$	−		−		+

B.

		-1		1	
$x+1$	−		+		+
$(x-1)^2$	+		+		+
$f(x)$	−		+		+

D.

		-1		1	
$(x+1)^2$	+		+		+
$(x-1)^3$	−		−		+
$f(x)$	−		−		+

Only A. and C. have correct sign of y.

A. $(1,1)$
B. $(1,2)$
C. $(1,3)$
D. $(2,3)$
E. $(2,1)$

6. $F(x) = f(g^3(x) + 1)$ and $f(0) = 8$, $f(9) = 7$, $g(0) = 2$, $f'(0) = 5$, $f'(9) = 2$ and $g'(0) = \frac{5}{6}$. Then $F'(0) =$

$F'(x) = \left(f'(g^3(x) + 1)\right)\left(3g^2(x) \cdot g'(x)\right)$

$F'(0) = \left(f'(g(0) + 1)\right)\left(3g^2(0)\ g'(0)\right)$

$= f'(2^3 + 1) \cdot \left(3 \cdot 2^2 \cdot \frac{5}{6}\right)$

$= f'(\ ) \cdot (10) = 20.$

A. 14
B. 15
C. 16
D. 20
E. 25

7. The functions $x(t)$ and $y(t)$ satisfy the equation $x^3 + y^3 = \frac{9}{2}xy$.
$x(1) = 2$, $y(1) = 1$ and $x'(1) = -4$. Then $y'(1) =$

Differentiating with respect to t :

$$\Rightarrow 3x^2 \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = \frac{9}{2}\left(\frac{dx}{dt} y + x\frac{dy}{dt}\right)$$

$t=1$ $$\Rightarrow 3 \cdot 2^2 \cdot (-4) + 3(1)^2 \frac{dy}{dt} = \frac{9}{2}\left(-4 \cdot 1 + 2 \cdot \frac{dy}{dt}\right)$$

$$\Rightarrow -48 + 3\frac{dy}{dx} = -18 + 9\frac{dy}{dx} \Rightarrow -6\frac{dy}{dx} = 30 \Rightarrow \frac{dy}{dx} = -5$$

A. -2

B. -3

C. -4

D. -5

E. -6

8. If $f(x) = \dfrac{x}{1+e^x}$ then $f'(1) =$

$$f'(x) = \frac{(1)(1+e^x) - (x)(e^x)}{(1+e^x)^2}$$

$$f'(1) = \frac{(1)(1+e) - (1)(e)}{(1+e)^2} = \frac{1}{(1+e)^2}$$

A. $\dfrac{1+2e}{(1+e)^2}$

B. $\dfrac{1}{(1+e)^2}$

C. $\dfrac{1-e}{(1+e)^2}$

D. $\dfrac{1-2e}{(1+e)^2}$

E. None of the above.

9. Let $F(x) = \int_0^x \sinh^3(t)dt$. Which of the following statements are true:

I. $F(x)$ is increasing
II. $F(1) = F(-1)$
III. $F(1) = -F(-1)$
IV. F has a minimum at $x = 0$

I. FALSE since $F'(x) = \sinh^3 x$ and $\sinh^3 x < 0$ for $x < 0$.

A. I and II.

B. I and III.

C. II and IV.

D. III and IV.

E. II and III.

10. After 5 days $\frac{1}{\sqrt{2}}$ of a sample of a radioactive element remains. The half-life of the element is

$A(t) = A(0)e^{kt}$

Want to solve $\frac{1}{2}A(0) = A(0)e^{kt}$ for t.

$\Rightarrow \frac{1}{2} = e^{kt} \to \ln\frac{1}{2} = kt \to t = \frac{1}{k}\ln\frac{1}{2}$ (*)

Need k to complete solution.

$A(5) = \frac{1}{\sqrt{2}}A(0) = A(0)e^{5k} \to \frac{1}{\sqrt{2}} = e^{k5}$

$\to 5k = \ln\frac{1}{\sqrt{2}} \to k = \frac{1}{5}\ln\frac{1}{\sqrt{2}}$

(*) $t = (\ln\frac{1}{2})\left(\frac{5}{\ln 1/\sqrt{2}}\right) = 5\,\frac{\ln 2^{-1}}{\ln 2^{-1/2}} = 5\,\frac{(-1)\ln 2}{(-\frac{1}{2})\ln 2} = 10.$

A. 2.5 days
B. 5 days
C. $\frac{5}{\ln 2}$ days
D. 10 days
E. $\frac{10}{\ln 2}$ days

11. If $f(x) = (\sqrt{x})^{e^{2x}}$ then $\frac{f'(x)}{f(x)} =$

Let $y = (\sqrt{x})^{e^{2x}}$.

Then $\ln y = (e^{2x})(\ln\sqrt{x}) = (e^{2x})(\frac{1}{2}\ln x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (2e^{2x})(\frac{1}{2}\ln x) + (e^{2x})(\frac{1}{2x})$

$\Rightarrow \frac{dy}{dx} = (\sqrt{x})^{e^{2x}}\left(e^{2x}\ln x + \frac{e^{2x}}{2x}\right)$

$\Rightarrow \frac{dy}{dx}\Big/f(x) = e^{2x}\ln x + \frac{e^{2x}}{2x} = e^{2x}\left(\ln x + \frac{1}{2x}\right)$

A. $e^{2x}\left[\ln x + \frac{1}{2x}\right]$
B. $\frac{1}{\sqrt{x}}e^{2x}$
C. $\ln x + 1$
D. $\frac{1}{\sqrt{x}} + e^{2x}$
E. $2e^{2x}$

12. $\int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{1+\sin^2 x}dx =$

$= \ln|1+\sin^2 x|\Big|_0^{\pi/2}$

$= \ln|1+(\sin\frac{\pi}{2})^2| - \ln|1+(\sin 0)^2|$

$= \ln 2$

A. $\frac{1}{2}$
B. $\ln 2$
C. 1
D. 2
E. None of the above.

13. The area between the graph of $y = x^2$ and $y = \sqrt{8x}$ is

A. $\int_0^2 (\sqrt{8x} - x^2)dx$

B. $\int_0^2 (x^2 - \sqrt{8x})dx$

C. $\int_0^1 (\sqrt{8x} - x^2)dx$

D. $\int_0^1 (x^2 - \sqrt{8x})dx$

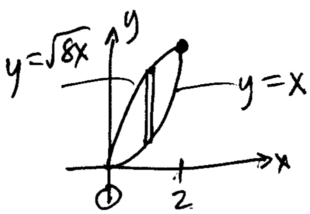
E. None of the above

Intersection of graphs:

$x^2 = \sqrt{8x} \rightarrow x^4 = 8x$

$\rightarrow x^4 - 8x = 0 \rightarrow x(x^3-8) = 0$

$\rightarrow x = 0, 2$



$$\text{Area} = \int_0^2 (\sqrt{8x} - x^2)\,dx$$

14. If $f(x) = x^2 \tan^{-1} x$ then $f'(1) =$

A. $\frac{\pi}{4}$

B. 1

C. $\frac{\pi}{2} + \frac{1}{2}$

D. $1 + \frac{\pi}{4}$

E. $\left(\frac{\pi}{4}\right)^2$

$$f'(x) = 2x \tan^{-1} x + x^2\left(\frac{1}{1+x^2}\right)$$

$$f'(1) = 2\tan^{-1} 1 + \frac{1}{2}$$

$$= 2 \cdot \frac{\pi}{4} + \frac{1}{2}$$

$$= \frac{\pi}{2} + \frac{1}{2}$$

15. $\int_0^1 7^x dx =$

A. 6

B. $6 \ln 7$

C. 7

D. $\frac{6}{\ln 7}$

E. $7 \ln 7$

$$= 7^x \frac{1}{\ln 7} \Big|_0^1$$

$$= \frac{1}{\ln 7}(7-1) = \frac{6}{\ln 7}$$

16. $\int_0^4 \frac{1}{16+x^2}dx = \frac{1}{16}\int_0^4 \frac{1}{\left(1+\left(\frac{x^2}{16}\right)\right)}dx$

$= \frac{1}{16}\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}dx = \frac{1}{16}\int_0^1 \frac{1}{1+u^2}4\,du$

let $u=\frac{x}{4}$, $du=\frac{1}{4}dx$

$u(0)=0$, $u(4)=1$

$= \frac{4}{16}\tan^{-1}u \Big|_0^1$

$= \frac{1}{4}(\tan^{-1}1 - \tan^{-1}0)$

$= \frac{1}{4}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{16}$

A. $\frac{\pi}{16}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

E. None of the above.

17. $\int \frac{\cosh x}{\sqrt{1-\sinh^2 x}}dx$

$= \sin^{-1}(\sinh x) + C$

A. $\sin^{-1}(\sinh(x)) + C$

B. $2\sqrt{1-\sinh^2 x} + C$

C. $\ln|\cosh x| + C$

D. $\sinh(\sqrt{1+x^2}) + C$

E. None of the above

18. If $F(x) = \int_{\sin x}^0 e^{t^2}\,dt$, then $F'(x) =$

$F(x) = -\int_0^{\sin x} e^{t^2}\,dt$

$F'(x) = -\left(e^{\sin^2 x}\right)(\cos x)$

A. $-2e^{\sin^2 x}\sin x$

B. $-e^{x^2}\cos x$

C. $-e^{\sin^2 x}$

D. $-e^{\cos^2 x}$

E. $-e^{\sin^2 x}\cos x$

19. $\lim_{x\to 0^-} \frac{\cos x}{\ln|1+x|} = \frac{1}{0^-} = -\infty$

A. $-\infty$

B. -1

C. 0

D. 1

E. ∞

note: $x \to 0^- \Rightarrow 1+x \to 1^-$

$\Rightarrow |1+x| \to 1^-$

$\Rightarrow \ln|1+x| \to 0^-$

20. Let $f''(x) = (x-1)x^2(x+1)^3$. Then the inflection points of f occur when

A. $x = 0$

B. $x = -1$

C. $x = -1, 1$

D. $x = -1, 0, 1$

E. f has no inflection points

	-1	0	1	
$x-1$	$-$	$-$	$-$	$+$
x^2	$+$	$+$	$+$	$+$
$(x+1)^3$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$-$	$-$	$+$
concave	UP	DOWN	DOWN	UP

inf. pt. inf. pt

21. Let $g'(x) = (x-1)x^4(x+1)^5$. The critical numbers of g are $x = -1, 0, 1$. g has

A. one relative maximum and two relative minima

B. one relative minimum and two relative maxima

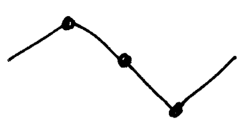
C. one relative minimum and one relative maximum

D. one relative maximum and no relative minimum

E. one relative minimum and no relative maximum

	-1	0	1	
$x-1$	$-$	$-$	$-$	$+$
x^4	$+$	$+$	$+$	$+$
$(x+1)^5$	$-$	$+$	$+$	$+$
$g'(x)$	$+$	$-$	$-$	$+$

graph of g



rel. max at $x=-1$

rel. min at $x=1$

22. $\displaystyle\lim_{x\to -1^+} \frac{|x|-1}{x+1} =$

A. -2

B. -1

C. 0

D. 1

E. does not exist

$x \to -1^+ \Rightarrow |x| - 1 = -x - 1$

$\therefore \displaystyle\lim_{x\to -1^+} \frac{|x|-1}{x+1} = \lim_{x\to -1^+} \frac{-x-1}{x+1}$

$= \displaystyle\lim_{x\to -1^+} -1 = -1$

23. $\int_{-1}^{1} x^3 \sin(x^4)dx =$

A. $2\cos 1$
B. $1+\cos(1)$
C. 0
D. $\cos(1)+\cos(-1)$
E. 2

Form is $\int \sin u \, du$.

Let $u = x^4$, $\rightarrow du = 4x^3 dx$, $u(-1) = u(1) = 1$.

$\int_{-1}^{1} x^3 \sin(x^4)\, dx = \int_{1}^{1} \sin u \cdot \frac{1}{4} du = 0$

24. Which of the following is a horizontal asymptote of $f(x) = \sqrt{x^2+4x+3} - x$

A. 0
B. 1
C. 2
D. 3
E. 4

Note: $\lim_{x\to\infty} \sqrt{x^2+4x+3} - x = \infty - \infty$.

Multiply by conjugate:

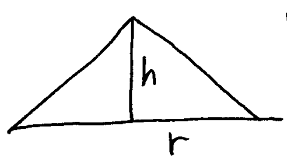
$(\sqrt{\quad} - x)\left(\frac{\sqrt{\quad} + x}{\sqrt{\quad} + x}\right) = \frac{x^2+4x+3-x^2}{\sqrt{x^2+4x+3} + x}$

Divide numerator and denominator by x;

$\frac{4+3/x}{\sqrt{1+4/x+3/x^2}+1} \rightarrow \frac{4}{1+1} = 2$ as $x \rightarrow \infty$

25. Sand is falling into a conical pile at a rate of 2 cubic feet per second. The height of the cone is always two-thirds of the radius of its base. Find the rate of change of the radius of the pile when it contains 6π cubic feet of sand. $(V = \frac{1}{3}\pi r^2 h)$

A. $\frac{\pi}{3}$
B. $\frac{\pi}{2}$
C. $\frac{1}{3\pi}$
D. $\frac{2}{3\pi}$
E. $\frac{1}{2}$



$h = \frac{2}{3} r$

Know: $\frac{dV}{dt} = 2$.

Want: $\frac{dr}{dt}$ at time when $V = 6\pi$.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \frac{2}{3} r = \frac{2\pi}{9} r^3$

$\Rightarrow \frac{dV}{dt} = \frac{2}{9}\pi 3r^2 \frac{dr}{dt} \Rightarrow 2 = \frac{2}{9}\pi 3 \cdot 3^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 2 \cdot \frac{9}{2} \cdot \frac{1}{\pi} \cdot \frac{1}{27}$

$= \frac{1}{3\pi}$

note: $6\pi = \frac{2\pi}{9} r^3 \rightarrow r^3 = 6\pi\left(\frac{9}{2\pi}\right) = 27 \rightarrow r = 3$