MATH 161 FALL 1999 FINAL EXAM

STUDENT NAME: SOLUTION KEY

STUDENT ID NUMBER: ______

RECITATION INSTRUCTOR: ______

INSTRUCTIONS:

1. Fill in your name, student ID number and division and section numbers on the mark-sense sheet. Also fill in the information requested above
2. This test booklet has 10 pages including this one. There are 20 questions, each worth 10 points.
3. Use a number 2 pencil to mark your choice of the correct answer in the spaces provided for questions 1-20 in the mark-sense sheet. Also show your work in this booklet.
4. Turn in BOTH the answer sheet and the question sheets to your instructor when you are finished.
5. No books, notes or calculators may be used.

1)

$$\lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x^2 - 5x} =$$

a) $\sqrt{5}$

b) $2\sqrt{5}$

c) $\frac{1}{\sqrt{5}}$

d) $\frac{1}{2\sqrt{5}}$

[✓ e) $\frac{1}{10\sqrt{5}}$]

$$= \lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x(x-5)}$$

$$= \lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x(\sqrt{x} - \sqrt{5})(\sqrt{x} + \sqrt{5})}$$

$$= \lim_{x \to 5} \frac{1}{x(\sqrt{x} + \sqrt{5})} = \frac{1}{5(\sqrt{5} + \sqrt{5})} = \frac{1}{10\sqrt{5}}$$

2)

$$\lim_{x \to 1^-} \frac{x^2 - 1}{|x - 1|} =$$

a) -1

b) 1

[✓ c) -2]

d) 2

e) 3

$$x \to 1^- \Rightarrow x < 1 \Rightarrow x - 1 < 0$$

$$\Rightarrow |x-1| = -(x-1)$$

$$\therefore \lim_{x \to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{-(x-1)}$$

$$= \lim_{x \to 1^-} \frac{x+1}{-1} = \frac{2}{-1} = -2$$

3)

$$\lim_{x\to-\infty}\frac{x-x^3}{1+2x^2}=$$

a) $\frac{1}{2}$

b) 1

c) -1

[✓ d) ∞]

e) $-\infty$

$$= \lim_{x\to-\infty}\frac{-x^3}{2x^2} = \lim_{x\to-\infty}\frac{-x}{2} = \infty$$

Alternate Method: divide numerator and denominator by x^3

$$\lim_{x\to-\infty}\frac{x-x^3}{1+2x^2} = \lim_{x\to-\infty}\frac{\frac{x}{x^3}-\frac{x^3}{x^3}}{\frac{1}{x^3}+\frac{2x^2}{x^3}}$$

$$= \lim_{x\to-\infty}\frac{\frac{1}{x^2}-1}{\frac{1}{x^3}+\frac{2}{x}} = \frac{-1}{0^-} = \infty$$

4) Which of the following statements are true?

I. If a function $f(x)$ is continuous at $x=a$ it is also differentiable at $x=a$.

II. If $f(x)$ is continuous for all x and $f'(a)=0$ then $f(a)$ is a relative maximum or minimum of f.

III. If $g(x)$ is differentiable and $g'(x)>0$ for x in (a,b), then the maximum of g on $[a,b]$ is $g(b)$.

[✓ a) Only III]

b) Only I and II

c) Only I and III

d) Only II and III

e) Only I

I. FALSE. Counter-example is $y=|x|$, which is continuous at $x=0$ but not differentiable at $x=0$.

II. FALSE. Counterexample is $y=x^3$. $y'=3x^2$ and $y'=0$ for $x=0$. Yet $(0,0)$ is not a relative extremum since $3x^2>0$ for $x\neq0$.

III. TRUE. $g'(x)>0 \Rightarrow g$ is increasing on (a,b) so maximum of g is at right endpoint b.

5) The derivative of $f(x) = \sin\left(4x^3 + \pi e^{4x} + \pi\cos(x)\right)$ at $x = 0$ is

a) π

b) 2π

c) 3π

[✓ d)] 4π

e) 5π

$$f'(x) = \left(\cos\left(4x^3 + \pi e^{4x} + \pi\cos(x)\right)\right)\left(12x^2 + 4\pi e^{4x} - \pi\sin(x)\right)$$

$$f'(0) = \left(\cos\left(0 + \pi + \pi\right)\right)\left(0 + 4\pi - 0\right)$$

$$= \left(1 \right)\left(4\pi \right)$$

$$= 4\pi.$$

6) The slope of the tangent line to the curve $x^2 - y^{3/2} = 1$ at $(3, 4)$ is

[✓ a)] 2

b) 3

c) $\frac{2}{3}$

d) 5

e) 8

Differentiate with respect to x

$$\Rightarrow \quad 2x - \frac{3}{2} y^{1/2} \frac{dy}{dx} = 0.$$

$$(x,y) = (3,4) \Rightarrow \quad 6 - \frac{3}{2} \cdot 2 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-6}{-3} = 2.$$

7) Capital, deposited in a bank at a fixed interest rate, will increase exponetially in value. Suppose the initial deposit of \$ 1,000 increases to \$ 1,500 in six years. How much will it be after 9 years?

a) $\sqrt{\frac{3}{2}}500$

[✓ b) $1000\left(\frac{3}{2}\right)^{3/2}$]

c) $1000^{3/2}$

d) $1500\ln\left(\frac{3}{2}\right)$

e) 1750

$A(t) = A(0)e^{kt} = 1000\,e^{kt}$

$A(9) = 1000\,e^{9k} \Rightarrow$ need to find k.

$A(6) = 1500 = 1000\,e^{6k} \rightarrow \frac{3}{2} = e^{6k}$

$\rightarrow \ln\frac{3}{2} = 6k \rightarrow k = \frac{1}{6}\ln\frac{3}{2}$

$\therefore A(9) = 1000\,e^{9(\frac{1}{6}\ln\frac{3}{2})}$

$= 1000\left(e^{\ln 3/2}\right)^{9/6}$

$= 1000\left(\frac{3}{2}\right)^{9/6} = 1000\left(\frac{3}{2}\right)^{3/2}$

8) A 5 foot ladder is leaning against a vertical wall. If the top of the ladder is falling at $\frac{1}{2}$ ft/sec, how fast is the bottom of the ladder moving away from the wall when its top is 3 feet above the floor?

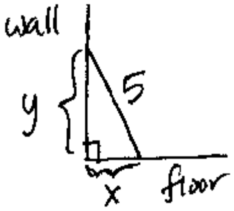
a) 2 ft/sec

b) $\frac{3}{2}$ ft/sec

c) 3 ft/sec

d) $\frac{3}{4}$ ft/sec

[✓ e) $\frac{3}{8}$ ft/sec]



Know: $\frac{dy}{dt} = -\frac{1}{2}$

want: $\frac{dx}{dt}$ at time when $y = 3$.

relate x and y : $x^2 + y^2 = 5^2$

differentiate w.r.t. t : $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

Substitute $y=3$, $x=4$, $\frac{dy}{dt} = -\frac{1}{2}$: $2(4)\left(\frac{dx}{dt}\right) + 2(3)\left(-\frac{1}{2}\right) = 0$

$\rightarrow \frac{dx}{dt} = \frac{-(3)(-\frac{1}{2})}{4} = \frac{3}{8}$

9) The minimum value of the sum of a positive number and four times its reciprocal is

- a) 1
- [✓ b) 4]
- c) 2
- d) $2\sqrt{2}$
- e) 6

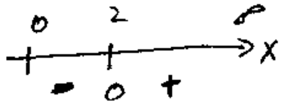
Let $x > 0$.

Let $S = x + 4\left(\frac{1}{x}\right)$.

Want minimum of S.

$$\frac{dS}{dx} = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = 0 \Rightarrow x = 2.$$

$\frac{dS}{dx}$ has sign of x^2-4



Thus minimum of S is $S(2)$

$= 2 + \frac{4}{2} = 4$,

∴ graph of S

10) The function $f(x) = x^x$ is defined for $x > 0$. The interval where f is increasing is

- a) $[1, \infty)$
- b) $[e^{-1}, e]$
- [✓ c) $[e^{-1}, \infty)$]
- d) $[e, \infty)$
- e) $(0, e]$

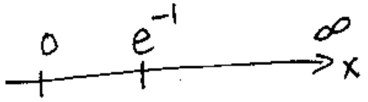
Let $y = x^x$. Then $\ln y = x \ln x$,

and $\frac{1}{y}\frac{dy}{dx} = (1)(\ln x) + x\left(\frac{1}{x}\right)$

$$\rightarrow \frac{dy}{dx} = (x^x)(\ln x + 1)$$

$\frac{dy}{dx}$ has sign of $\ln x + 1$ since x^x is positive.

$\ln x + 1 = 0 \rightarrow \ln x = -1 \rightarrow x = e^{-1}$



$\ln x + 1$ $-$ 0 $+$

y dec. inc. $\Rightarrow$ y increasing on $[e^{-1}, \infty)$

11) The derivative of a function $f(x)$ is given by

$$f'(x) = (x-2)^2(x-1)(x+1)^3(x-3)^4.$$

Which of the following is correct?

a) $f(2)$ and $f(3)$ are relative maxima, $f(1)$ and $f(-1)$ are relative minima.

b) $f(2)$ and $f(3)$ are relative minima, $f(1)$ and $f(-1)$ are relative maxima.

c) $f(1)$ and $f(-1)$ are relative maxima, $f(2)$ and $f(3)$ are neither.

[✓ d)] $f(-1)$ is a relative maximum, $f(1)$ is a relative minimum, $f(2)$ and $f(3)$ are neither.

e) $f(-1)$ is a relative minimum, $f(1)$ is a relative maximum, $f(2)$ and $f(3)$ are neither.

We want to analyze the sign of $f'(x)$.

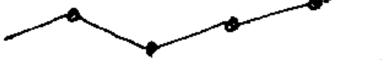
$-\infty$ -1 1 2 3 ∞ x

$(x-2)^2$	+	+	+	+	+
$x-1$	−	−	+	+	+
$(x+1)^3$	−	+	+	+	+
$(x-3)^4$	+	+	+	+	+
$f'(x)$	+	−	+	+	+

rel min at $x=1$

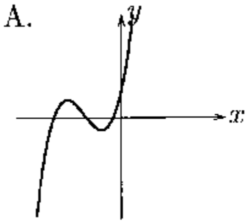
rel max at $x=-1$

graph of f

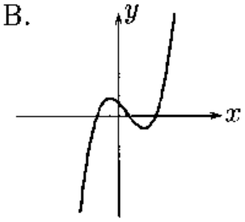


12) Which of the following looks most like the graph of $f(x) = 3x^3 + 4x^2 + 3x + 2$.

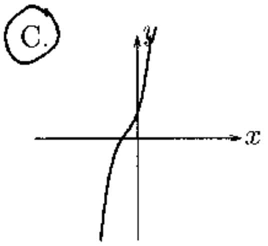
A.



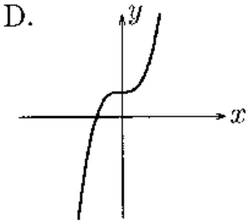
B.



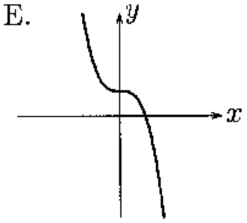
[✓ C.]



D.



E.



1. For large x, $f(x) \approx 3x^3$ so choice E is eliminated,

2. $f'(x) = 9x^2 + 8x + 3 = 0 \rightarrow x = \dfrac{-8 \pm \sqrt{64-108}}{18} \rightarrow$ no real solutions $\rightarrow$ no relative extrema $\rightarrow$ A and B eliminated.

3. $f''(x) = 18x + 8 = 0 \rightarrow x = -\frac{4}{9}$ and $\overset{-4/9}{\longrightarrow}$ $f''(x)$ $-$ 0 $+$

$\rightarrow$ inflection point at $x = -\frac{4}{9} \rightarrow$ C is correct.

13)

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx =$$

- a) **1**
- b) $\frac{3}{2}$
- c) $\frac{5}{3}$
- [✓ d)] $\frac{8}{3}$
- e) $\frac{5}{2}$

Use substitution: Let $u = x+1$.

Then $du = dx$, $x = u-1$, $u(0) = 1$, $u(3) = 4$.

Substituting, $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^4 \frac{u-1}{\sqrt{u}}\,du$

$$= \int_1^4 \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_1^4$$

$$= \left(\frac{2}{3}\cdot 8 - 2\cdot 2\right) - \left(\frac{2}{3}\cdot 1 - 2\cdot 1\right)$$

$$= 8/3$$

14)

$$\int_0^1 \frac{d}{dx}\ln(1+x^6)\,dx =$$

- a) $\frac{1}{2}$
- b) $\ln(1+\sqrt{2})$
- [✓ c)] $\ln(2)$
- d) $\frac{\ln(3)}{2}$
- e) $\frac{\ln(1+\sqrt{2})}{2}$

$$= \ln(1+x^6)\Big|_0^1$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

15)

$$\int_0^1 \frac{7^x}{1+7^x}\,dx =$$

[✓ a) $\frac{\ln 4}{\ln 7}$]
b) $\frac{\ln 3}{\ln 7}$
c) $\frac{1}{\ln 7}$
d) $\ln 7$
e) $2\ln 7$

$$= \frac{1}{\ln 7} \ln\left(1+7^x\right)\Big|_0^1$$

$$= \frac{1}{\ln 7}\left(\ln 8 - \ln 2\right)$$

$$= \frac{1}{\ln 7}\left(3\ln 2 - \ln 2\right)$$

$$= \frac{1}{\ln 7}\left(2\ln 2\right)$$

$$= \frac{1}{\ln 7}\ln 4 = \frac{\ln 4}{\ln 7}$$

16)

$$\int_0^{\frac{1}{2\sqrt{2}}} \frac{1}{\sqrt{1-4x^2}}dx =$$

a) $\frac{\pi}{3}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{6}$
d) $\frac{\pi}{7}$
[✓ e) $\frac{\pi}{8}$]

Let $u = 2x$, then $du = 2dx \rightarrow dx = \frac{1}{2}du$

and $u(0) = 0$ and $u\left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

Substituting, $\int_0^{\frac{1}{2\sqrt{2}}} \frac{1}{\sqrt{1-(2x)^2}}\,dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-u^2}}\,\frac{1}{2}\,du$

$$= \frac{1}{2}\sin^{-1}u\,\Big|_0^{1/\sqrt{2}} = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

17)

$$\int_0^{\frac{1}{2}} \frac{2\,dx}{1+4x^2} =$$

[✓ a)] $\frac{\pi}{4}$

b) $\frac{\pi}{6}$

c) $\frac{\pi}{8}$

d) $\frac{\pi}{10}$

e) $\frac{\pi}{5}$

Let $u = 2x$. Then $du = 2\,dx$, $u(0) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

Substituting, $\displaystyle\int_0^{\frac{1}{2}} \frac{2\,dx}{1+(2x)^2} = \int_0^1 \frac{du}{1+u^2}$

$$= \tan^{-1} u \Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

18) What is the value of

$$\frac{d}{dx}\int_1^{3x^2+2} \frac{1}{\sqrt{t^2+1}}\,dt \text{ at } x = 1?$$

a) $\frac{2}{\sqrt{3}}$

[✓ b)] $\frac{6}{\sqrt{26}}$

c) $\frac{4}{\sqrt{8}}$

d) $\frac{2}{\sqrt{26}}$

e) $\frac{2}{\sqrt{27}}$

$$\frac{d}{dx}\int_1^{3x^2+2} \frac{1}{\sqrt{t^2+1}}\,dt = \frac{1}{\sqrt{(3x^2+2)^2+1}}\,(6x)$$

$$x=1 \Rightarrow \frac{1}{\sqrt{5^2+1}}\cdot 6 = \frac{6}{\sqrt{26}}$$

19) If $f(x) = x^5 + x^3$, then $\left(f^{-1}\right)'(2)$ is equal to

a) 1

b) $5^4 + 35^2$

c) 2

d) 8

[✓ e) $\frac{1}{8}$]

Note: $f(1) = 1^5 + 1^3 = 2$.

$$(f^{-1})'(2) = \frac{1}{f'(1)}.$$

$f'(x) = 5x^4 + 3x^2$. $f'(1) = 5+3 = 8$.

$$\therefore (f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{8}.$$

20) The area of the region bounded by the curves $y = x^2 + 1$ and $y = -2x^2 + 13$ is

a) 6

[✓ b) 32]

c) 20

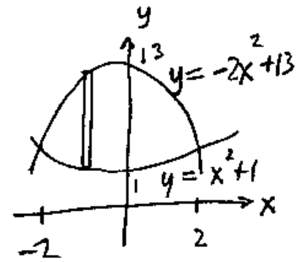
d) 12

e) 40

intersection of curves: $x^2 + 1 = -2x^2 + 13$

$\rightarrow 3x^2 = 12 \rightarrow x = \pm 2$.

Region:



typical slice has area $\cong \left(-2x^2 + 13 - (x^2+1)\right)dx$

$$\text{Area of region} = \int_{-2}^{2} \left(-2x^2 + 13 - (x^2+1)\right) dx$$

$$= \int_{-2}^{2} \left(-3x^2 + 12\right) dx = \left(-x^3 + 12x\right)\Big|_{-2}^{2}$$

$$= (-8+24) - (8-24) = 16 - (-16) = 32.$$