MATH 161 – 161E – SPRING 1998 – SECOND EXAM – March 19, 1998

STUDENT NAME: SOLUTION KEY

STUDENT ID: ______________________________

RECITATION INSTRUCTOR: ______________________________

INSTRUCTIONS:

1. This test booklet has 6 pages including this one.

2. Fill in your name, your student ID number, and your recitation instructor's name above.
3. Use a number 2 pencil on the mark-sense sheet (answer sheet).
4. On the mark-sense sheet, fill in the recitation instructor's name and the course number.
5. Fill in your name and student ID number, blacken the appropriate spaces, and sign the mark-sense sheet.
6. Mark the division and section number of your class and blacken the corresponding circles, including the circles for the zeros. If you do not know your division and section number ask your instructor.
7. There are 15 questions, each worth 7 points. Blacken your choice of the correct answer in the spaces provided for questions 1-15. Turn in BOTH the answer sheet and the question sheets to your instructor when you are finished.
8. No books, notes or calculators may be used.

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1) Given $f(x) = \dfrac{x}{x^2-1}$, $f''(x)$ is

a) $-\dfrac{1+x^2}{(x^2-1)^4}$

b) $\dfrac{2x(x^2+3)}{(x^2-1)^3}$

c) $\dfrac{1+x^2}{(x^2-1)^4}$

d) $\dfrac{2x(x^2+3)}{(x^2-1)^2}$

e) $\dfrac{3x-2}{(x^2-1)^4}$

$$f'(x) = \frac{(1)(x^2-1)-(x)(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2}$$

$$f''(x) = \frac{(-2x)(x^2-1)^2-(-1-x^2)(2)(x^2-1)(2x)}{(x^2-1)^4}$$

$$= \frac{(-2x)(x^2-1)-(-1-x^2)(2)(2x)}{(x^2-1)^3}$$

$$= \frac{-2x^3+2x+4x+4x^3}{(x^2-1)^3}$$

$$= \frac{2x^3+6x}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

2) Given that y is a function of x and that $x^2 + \frac{x}{y} = -2$, then $y'(x)$ at the point $(1, -\frac{1}{3})$ is

a) $\frac{4}{9}$

b) -9

c) 12

d) $\frac{-1}{9}$

e) Impossible to determine

$$2x + \frac{(1)(y) - x\frac{dy}{dx}}{y^2} = 0$$

$$(1, -\tfrac{1}{3}) \Rightarrow 2 + \frac{-\frac{1}{3} - \frac{dy}{dx}}{\frac{1}{9}} = 0 \Rightarrow 2 + \left(-3 - 9\frac{dy}{dx}\right) = 0$$

$$\Rightarrow -9\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{9}.$$

3) A beacon on a lighthouse 1 mile from shore revolves at the rate of 8 radians/min. Assuming that the shoreline is straight, calculate the speed at which the spotlight is sweeping across the shoreline as it lights up the sand 3 miles from the lighthouse.

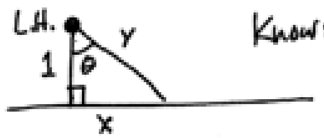
a) 24 miles/min

b) 10 miles/min

c) 50π miles/min

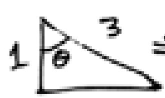
d) 72 miles/min

e) $\frac{32}{5}\pi$ miles/min



Know: $\frac{d\theta}{dt} = 8$ want: $\frac{dx}{dt}$ when $y = 3$

$$\tan\theta = x \rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt} \rightarrow (3^2)(8) = \frac{dx}{dt}$$



$\Rightarrow \sec\theta = 3$

$$\rightarrow \frac{dx}{dt} = 72$$

4) Using linear approximation at $x = 16$, we find that $(17)^{1/4}$ is approximately equal to

a) $\frac{63}{32}$

b) $\frac{64}{32}$

c) $\frac{66}{32}$

d) $\frac{65}{32}$

e) $\frac{67}{32}$

Let $f(x) = x^{\frac{1}{4}} \Rightarrow f'(x) = \frac{1}{4}x^{-3/4}$

$$(17)^{\frac{1}{4}} = (16+1)^{\frac{1}{4}} = f(16) + f'(16)(dx)$$
$$= 16^{\frac{1}{4}} + \frac{1}{4}(16)^{-3/4}(1)$$
$$= 2 + \frac{1}{4}\cdot\frac{1}{8}$$
$$= 2 + \frac{1}{32}$$
$$= \frac{64+1}{32} = \frac{65}{32}$$

5) A radio tower is 130 ft. tall and has been assembled on the ground lying on its side. A motorized device raises its top at a constant rate of 10 ft/min, keeping the base of the tower fixed with respect to the ground. At what rate is the distance from the top of the tower to the vertical position changing when the top of the tower is 120 ft from the ground?

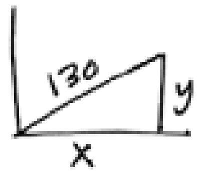
a) 24 ft/min

b) 50 ft/min

c) −10 ft/min

d) −50 ft/min

e) −24 ft/min



know: $\frac{dy}{dt} = 10$ want: $\frac{dx}{dt}$ when $y = 120$

$x^2 + y^2 = 130^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\rightarrow 2(50)\frac{dx}{dt} + 2(120)(10) = 0 \rightarrow \frac{dx}{dt} = \frac{(-120)(10)}{50} = -24$

note: $y = 120 \rightarrow x = 50$

6) A ferris wheel is 100 ft in radius and revolves at a rate of 1.5 radians/min. How fast is a passenger rising when she is 60 ft higher than the center of the ferris wheel and rising ?

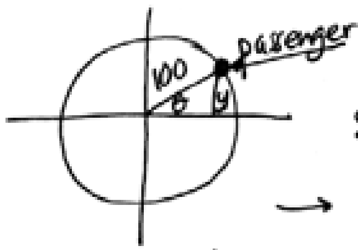
a) 40 ft/min

b) 90 ft/min

c) 120 ft/min

d) 30π ft/min

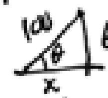
e) 200 ft/min



know: $\frac{d\theta}{dt} = 1.5$ want: $\frac{dy}{dt}$ when $y = 60$

$\sin\theta = \frac{y}{100} \rightarrow \cos\theta\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt}$

$\rightarrow \frac{8}{10}(1.5) = \frac{1}{100}\frac{dy}{dt} \rightarrow \frac{dy}{dt} = 100\left(\frac{8}{10}\right)(1.5) = 120.$

note: 

$\left(\Rightarrow x = \sqrt{100^2 - 60^2} = \sqrt{6400} = 80\right)$ $\left(\cos\theta = \frac{80}{100} = \frac{8}{10}\right)$

7) There are two critical numbers of $f(x) = (x^2 + 2x)e^{-x}$. Their product is

a) −2

b) −1

c) 0

d) 1

e) 2

$f'(x) = (2x+2)e^{-x} + (x^2+2x)(e^{-x})(-1)$

$= e^{-x}(2x + 2 - x^2 - 2x)$

$= e^{-x}(2 - x^2) = 0 \rightarrow x = \pm\sqrt{2}$

product of critical pts $= (\sqrt{2})(-\sqrt{2}) = -2$

8) If we use the Newton-Raphson method to find an approximate solution to $x^3 - 2x - 5 = 0$ and we start with $c_1 = 2$, then c_2 is equal to

a) $\frac{17}{10}$

b) $\frac{19}{10}$

c) $\frac{21}{10}$

d) $\frac{23}{10}$

e) $\frac{25}{10}$

Let $f(x) = x^3 - 2x - 5$. Then $f'(x) = 3x^2 - 2$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2 - \frac{8-4-5}{12-2} = 2 - \frac{-1}{10}$$

$$= 2 + \frac{1}{10} = \frac{21}{10}$$

9) The maximum value of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 10 \quad \text{in} \quad [1,3] \quad \text{is}$$

a) -10

b) -1

c) 1

d) 12

e) 17

critical pts: $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$
$= 6(x-2)(x+1) = 0 \rightarrow x = -1, 2$

note: $x = -1$ is not in the interval $[1,3]$

x	$f(x) = 2x^3 - 3x^2 - 12x + 10$
1	$f(1) = 2 - 3 - 12 + 10 = -3$
2	$f(2) = 16 - 12 - 24 + 10 = -12$
3	$f(3) = 54 - 27 - 36 + 10 = 1 \longleftarrow$ MAX

10) The minimum length of a fence built to enclose a rectangular region of 1250 square miles with one side using a river as a natural boundary which does not need to be fenced is

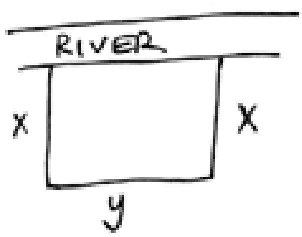
a) 75 miles

b) 100 miles

c) 125 miles

d) 150 miles

e) 175 miles



$xy = 1250 \rightarrow y = \frac{1250}{x}$

want minimum of $F = 2x + y$

$F(x) = 2x + \frac{1250}{x} \rightarrow F'(x) = 2 - \frac{1250}{x^2}$

(domain: $x > 0$)

$= \frac{2x^2 - 1250}{x^2}$

$= 0 \rightarrow x^2 = \frac{1250}{2} = 625$

$\rightarrow x = 25$

sign of $F'(x)$: on $(0, \infty)$, with 25 marked

$2x^2 - 1250$: $-$ 0 $+$

F: min. at 25

$F(25) = 2(25) + \frac{1250}{25} = 50 + 50 = 100$

11) Find all numbers c in $(0,1)$ for which the line tangent to the graph of $f(x) = x^3 - ax + b$ at $(c, f(c))$ is parallel to the line segment joining $(0, f(0))$ and $(1, f(1))$

a) $c = \frac{1}{\sqrt{3}}$

b) $c = \frac{1}{2}$

c) $c = \frac{1}{\sqrt{2}}$

d) $c = \frac{1}{3}$

e) c depends of the constants a and b

By the Mean Value Theorem: $f'(c) = \frac{f(0) - f(1)}{0-1}$, $0<c<1$.

$f'(x) = 3x^2 - a \rightarrow 3c^2 - a = \frac{(b) - (1-a+b)}{-1}$

$\rightarrow 3c^2 - a = 1 - a$

$\rightarrow 3c^2 = 1$

$\rightarrow c^2 = \frac{1}{3} \rightarrow c = \frac{1}{\sqrt{3}}$ (note: $0<c<1$)

12) Let $F(x)$ satisfy $F'(x) = \cos(2x)$ and $F(0) = 1$. Then $F(\frac{\pi}{4})$ is equal to

a) $\frac{1}{2}$

b) $\frac{3}{2}$

c) 1

d) 2

e) $\frac{2}{3}$

$F'(x) = \cos 2x \rightarrow F(x) = \frac{1}{2}\sin 2x + C$

$F(0) = \frac{1}{2}\sin 0 + C = C$ and $F(0) = 1$

Therefore, $C = 1$ and $F(x) = \frac{1}{2}\sin 2x + 1$

Hence, $F(\frac{\pi}{4}) = \frac{1}{2}\sin\frac{\pi}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$

13) The function $f(x) = 3x^5 - 5x^3 + 100$ is

a) is increasing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$

b) increasing in $(-\infty, -1)$, and decreasing in $(-1, \infty)$

c) increasing in $(-\infty, 0)$, and decreasing in $(0, \infty)$

d) increasing in $(-\infty, 2)$, and decreasing in $(2, \infty)$

e) always increasing.

$f'(x) = 15x^4 - 15x^2$

$= 15x^2(x^2 - 1)$

$= 0 \rightarrow x = -1, 0, 1$

	$-\infty$ to -1	-1 to 0	0 to 1	1 to ∞
$15x^2$	+	+	+	+
x^2-1	+	−	−	+
$f'(x)$	+	−	−	+

sign of $f'(x)$

f

14) Suppose you have a cache of a radioactive material whose half life is 2000 years. The length of time that it would take for one fifth of the material to disappear is

a) 400 years

b) $2000\dfrac{\ln(5)-\ln(2)}{\ln(2)}$ years

c) 800 years

d) $1000\dfrac{\ln(5)-\ln(4)}{\ln(2)}$ years

e) $2000\dfrac{\ln(5)-\ln(4)}{\ln(2)}$ years

Find k: $P(2000) = \frac{1}{2}P(0) = P(0)e^{2000k}$

$\to \frac{1}{2} = e^{2000k} \to \ln\frac{1}{2} = 2000k \to k = \dfrac{\ln\frac{1}{2}}{2000}$

Solve for t: $P(t) = \frac{4}{5}P(0) = P(0)e^{\left(\frac{1}{2000}\ln\frac{1}{2}\right)t}$

$\to \frac{4}{5} = e^{\frac{\ln 1/2}{2000}t} \to \ln\frac{4}{5} = \left(\frac{1}{2000}\ln\frac{1}{2}\right)t$

$\to t = 2000\dfrac{\left(\ln\frac{4}{5}\right)}{(-\ln 2)} = 2000\left(\dfrac{\ln 4-\ln 5}{-\ln 2}\right)$

15) The relative extreme values of $f(x) = x^4 - 2x^3 + x^2$ are

a) Three minima at $x = \frac{1}{2}$, $x = 1$ and $x = 0$,

b) Three maxima at $x = \frac{1}{2}$, $x = 1$ and $x = 0$,

c) One maximum at $x = 1$, two minima at $x = \frac{1}{2}$ and $x = 0$,

d) One maximum at $x = 0$, two minima at $x = \frac{1}{2}$ and $x = 1$,

e) One maximum at $x = \frac{1}{2}$, two minima at $x = 0$ and $x = 1$,

$$f'(x) = 4x^3 - 6x^2 + 2x$$
$$= 2x(2x^2 - 3x + 1)$$
$$= 2x(2x-1)(x-1)$$
$$= 0 \longrightarrow x = \tfrac{1}{2}, 1, 0$$

$-\infty \quad 0 \quad \frac{1}{2} \quad 1 \quad \infty \quad x$

$2x$	−	+	+	+
$2x-1$	−	−	+	+
$x-1$	−	−	−	+
$f'(x)$	−	+	−	+

f