MATH 161 - 161E – SPRING 1998 – FINAL EXAM – April 16, 1998

STUDENT NAME: SOLUTION KEY

STUDENT ID: ____________________

RECITATION INSTRUCTOR: ____________________

INSTRUCTIONS:

1. This test booklet has 11 pages including this one.
2. Fill in your name, your student ID number and your recitation instructor's name above.
3. Use a number 2 pencil on the mark-sense sheet (answer sheet).
4. On the mark-sense sheet, fill in the recitation instructor's name and the course number.
5. Fill in your name and student ID number, blacken the appropriate spaces, and sign the mark-sense sheet.
6. Mark the division and section number of your class and blacken the corresponding circles, including the circles for the zeros. If you do not know your division and section number ask your instructor.
7. There are 20 questions, each worth 10 points. Blacken your choice of the correct answer in the spaces provided for questions 1-20. Turn in BOTH the answer sheet and the question sheets to your instructor when you are finished.
8. No books, notes or calculators may be used.

1.

$$\lim_{x\to 0} \quad x\cos\left(\frac{1}{x^2}\right), \qquad \text{is equal to}$$

a) Does not exist

b) 1

c) ∞

d) $-\infty$

e) 0

Use the squeeze theorem:

for all $x \neq 0$, $-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

for $x > 0$, $-x \le x\cos(1/x^2) \le x$

for $x < 0$, $-x \ge x\cos(1/x^2) \ge x$.

both $\lim_{x\to 0^-} x$ and $\lim_{x\to 0^+} x = 0$, and therefore $\lim_{x\to 0} x\cos\frac{1}{x^2} = 0$.

2.

$$\lim_{x\to 4^-} \frac{|x-4|(x-3)^2}{x-4} \quad \text{is equal to}$$

a) 1

b) −1

c) ∞

d) $-\infty$

e) Cannot be determined.

$x \to 4^- \Rightarrow x < 4$

$\Rightarrow x-4 < 0$

$\Rightarrow |x-4| = -(x-4)$

Thus, $\displaystyle\lim_{x\to 4^-} \frac{|x-4|(x-3)^3}{x-4} = \lim_{x\to 4^-} \frac{-(x-4)(x-3)^3}{x-4}$

$\displaystyle = \lim_{x\to 4^-} -(x-3)^3 = -(4-3)^3 = -1$

3. Let $f(x) = \begin{cases} \dfrac{x^2+4x-5}{x+5}, & x \neq -5 \\ a, x = -5. \end{cases}$ If f is continuous at $x = -5$, then a is equal to

a) 2

b) −3

c) −4

d) 4

e) −6

want: $\displaystyle\lim_{x\to -5} f(x) = f(-5)$.

note that $f(-5) = a$

and $\displaystyle\lim_{x\to -5} f(x) = \lim_{x\to -5} \frac{x^2+4x-5}{x+5}$

$\displaystyle = \lim_{x\to -5} \frac{(x+5)(x-1)}{x+5}$

$\displaystyle = \lim_{x\to -5} (x-1)$

$= -5-1 = -6$

$\therefore a = -6$

4. Given $f(x) = \frac{x^2}{2}\sin(x)$, $f'(x)$ is equal to

   a) $x\sin(x) + \frac{x^2}{2}\cos(x)$

   b) $\frac{x}{2}\sin(x) + \frac{x^2}{2}\cos(x)$

   c) $x\sin(x) - \frac{x^2}{2}\cos(x)$

   d) $\frac{x}{2}\sin(x) - \frac{x^2}{2}\cos(x)$

   e) $x\cos(x)$

$$f'(x) = (x)(\sin x) + \left(\frac{x^2}{2}\right)(\cos x)$$

5. Given $f(x) = \sin(\pi\cos(x))$, $f''(0)$ is equal to

   a) $-\pi$

   b) -2π

   c) 2π

   d) π

   e) $\cos\left(\frac{\pi}{8}\right)$

$$f'(x) = \Big(\cos\big(\pi\cos(x)\big)\Big)\big(-\pi\sin x\big)$$

$$f''(x) =$$

$$\Big(\big(-\sin(\pi\cos x)\big)(-\pi\sin x)\Big)(-\pi\sin x) + \big(\cos(\pi\cos x)\big)(-\pi\cos x)$$

$$\therefore\ f''(0) = (-\sin\pi)(-\pi\sin 0)(-\pi\sin 0) + (\cos\pi)(-\pi\cos 0)$$

$$= 0 + (-1)(-\pi(1))$$

$$= \pi.$$

6. Which of the following is an equation of the line perpendicular to the graph of $f(x) = x^3 + 2x + 3$ at the point $(1, 6)$?

   a) $3y + x - 19 = 0$

   b) $5y + x - 31 = 0$

   c) $2y - x - 11 = 0$

   d) $5y - 10x - 20 = 0$

   e) There is no line perpendicular to the graph of f at this point.

Slope of tangent line is $f'(x) = 3x^2 + 2$; $f'(1) = 5$.

Slope of perpendicular line is $-\frac{1}{5}$

Perpendicular Line : $y - 6 = -\frac{1}{5}(x-1)$

$\Rightarrow 5y - 30 = -x + 1 \Longrightarrow 5y + x - 31 = 0$

7. A spherical balloon is inflated at the rate of 4 cubic centimeters per second. When the radius is 5 centimeters, the radius is increasing at the rate $\left(V = \frac{4}{3}\pi r^3\right)$

   a) $\frac{1}{75\pi}$

   b) $\frac{3}{100\pi}$

   c) $\frac{1}{25\pi}$

   d) $\frac{3}{50\pi}$

   e) $\frac{3}{25\pi}$

Know: $\frac{dV}{dt} = 4$

Want: $\frac{dr}{dt}$ at time when $r = 5$.

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow 4 = 4\pi 5^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{4}{4\pi 5^2} = \frac{1}{25\pi}$.

8. The minimum length of a fence built to enclose a rectangular region of 2400 square miles with one side and half of another, which are perpendicular to each other, using a river as a natural boundary which does not need to be fenced is

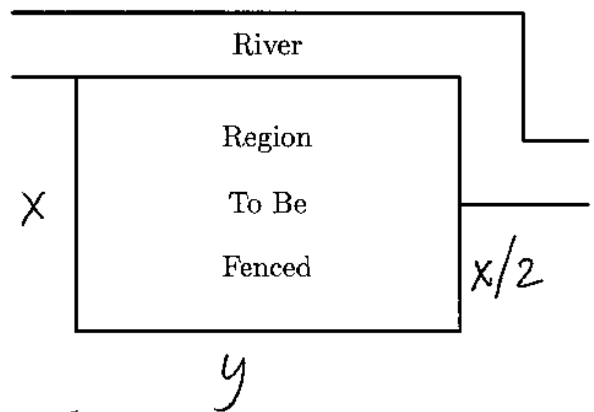
   a) 75 miles

   b) 100 miles

   c) 120 miles

   d) 130 miles

   e) 150 miles



Want: Minimum of $F = \frac{3}{2}x + y$,
where $xy = 2400$.

$\Rightarrow F(x) = \frac{3}{2}x + \frac{2400}{x}, \quad x > 0.$

$F'(x) = \frac{3}{2} - \frac{2400}{x^2} = \frac{3x^2 - 4800}{2x^2} = 0 \rightarrow x^2 = \frac{4800}{3} = 1600$

$\rightarrow x = 40$

0 40 ∞ x

$3x^2 - 4800$: − 0 +

graph of F: min.

$F(40) = \frac{3}{2}(40) + \frac{2400}{40} = 120.$

9. A colony of bacteria is growing exponentially. If 12 hours are required for the number of bacteria to grow from 4000 to 6000, then the time (in hours) required for their number to grow from 7000 to 14000 is

   a) impossible to determine

   b) $\dfrac{12\ln(2)}{\ln(3) - \ln(2)}$

   c) $\dfrac{7\ln(3)}{\ln(2)}$

   d) $\dfrac{14\ln(2)}{\ln(3)}$

   e) $\dfrac{3\ln(14)}{2}$

Want: doubling time, i.e., t that satisfies $2A(0) = A(0)e^{kt}$

$\Rightarrow 2 = e^{kt} \rightarrow \ln 2 = kt$

$* \Rightarrow t = \frac{1}{k}\ln 2 \Rightarrow$ need to find k

Let $A(0) = 4000$, Then $A(12) = 6000$.

Therefore $A(12) = 6000 = 4000e^{12k}$

$\rightarrow \frac{3}{2} = e^{12k} \rightarrow \ln\frac{3}{2} = 12k \rightarrow k = \frac{1}{12}\ln\frac{3}{2}$

Therefore $* \Rightarrow t = 12\dfrac{\ln 2}{\ln\frac{3}{2}} = 12\dfrac{\ln 2}{\ln 3 - \ln 2}$

10. If we use the Newton-Raphson method to find an approximate solution to $x^3-2x+3=0$ and we start with $c_1=-1$, then c_2 is equal to

a) -2

b) -3

c) -4

d) -5

e) -1

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = c_1 - \frac{c_1^3-2(c_1)+3}{3c_1^2-2}$$

$$= (-1) - \frac{(-1)^3-2(-1)+3}{3(-1)^2-2}$$

$$= -1 - \frac{4}{1}$$

$$= -5$$

11. Let M and m be the maximum and the minimum values of $f(x)=x^2-2\ln(x)$ in $[1/2, 2]$ respectively. Then $M-m$ is

a) $-\frac{3}{4}+\ln(4)$

b) $\frac{3}{4}+\ln(4)$

c) $\frac{15}{4}$

d) $\frac{15}{4}-2\ln(4)$

e) $3-\ln(4)$

$$f'(x) = 2x - 2\cdot\frac{1}{x} = \frac{2x^2-2}{x} = 0$$

$$\to x = \pm 1$$

x	$x^2-2\ln x$
$\frac{1}{2}$	$\frac{1}{4}-2\ln\frac{1}{2} = \frac{1}{4}+2\ln 2 = \frac{1}{4}+\ln 4 > 1$
1	$1-0=1$
2	$4-2\ln 2 = 4-\ln 4$

$f(1)=1$ is min $\qquad f(2)=4-\ln 4$ is max

$$\max - \min = 4-\ln 4 - 1 = 3-\ln 4$$

12. The function $f(x) = 5x^5 - 3x^3 + 1$ has

a) a maximum at $x = -\frac{3}{5}$ a minimum at $x = \frac{3}{5}$ and an inflection point at $(0, f(0))$

b) a minimum at $x = -\frac{3}{5}$ a maximum at $x = -\frac{3}{5}$ and an inflection point at $(0, f(0))$

c) an inflection point at $\left(-\frac{3}{5}, f(-\frac{3}{5})\right)$ a maximum at $x = 0$ and a minimum at $x = \frac{3}{5}$

d) a maximum at $x = -\frac{3}{5}$ and two minima, one at $x = 0$ the other at $x = \frac{3}{5}$

e) three inflection points at $\left(-\frac{3}{5}, f\left(-\frac{3}{5}\right)\right)$, $(0, f(0))$ and $\left(\frac{3}{5}, f\left(\frac{3}{5}\right)\right)$

$f'(x) = 25x^4 - 9x^2 = x^2(25x^2 - 9)$

x: $-\infty$ … ∞	$-\infty$ to $-\frac{3}{5}$	$-\frac{3}{5}$ to 0	0 to $\frac{3}{5}$	$\frac{3}{5}$ to ∞
x^2	+	+	+	+
$25x^2-9$	+	−	−	+
$f'(x)$	+	−	−	+

f 

$f''(x) = 100x^3 - 18x = x(100x^2 - 18)$

x: $-\infty$ … ∞	$-\infty$ to $-\sqrt{\frac{18}{100}}$	$-\sqrt{\frac{18}{100}}$ to 0	0 to $\sqrt{\frac{18}{100}}$	$\sqrt{\frac{18}{100}}$ to ∞
x	−	−	+	+
$100x^2-18$	+	−	−	+
$f''(x)$	−	+	−	+

inflection pts at $x = 0, \pm\sqrt{\frac{18}{100}}$.

13.

$$\int_{\ln 3}^{\ln 8} e^t \sqrt{1 + e^t}\, dt \quad \text{is equal to}$$

a) 8

b) $\frac{39}{105}$

c) $\frac{8}{5}$

d) $\frac{38}{3}$

e) $\frac{8\ln(3) - 8}{5}$

$$= \frac{2}{3}(1 + e^t)^{3/2} \Big|_{\ln 3}^{\ln 8}$$

$$= \frac{2}{3}\left(\left(1 + e^{\ln 8}\right)^{3/2} - \left(1 + e^{\ln 3}\right)^{3/2}\right)$$

$$= \frac{2}{3}\left((9)^{3/2} - (4)^{3/2}\right)$$

$$= \frac{2}{3}(27 - 8) = \frac{2}{3}(19) = \frac{38}{3}$$

14. If

$$f(x) = \int_0^{x^2} \sqrt{9+t^2}dt \quad \text{then} \quad f'(2) =$$

a) $\frac{40}{3}$

b) $4\sqrt{13} - 3$

c) 20

d) 17

e) $4\sqrt{13}$

$$f'(x) = \left(\sqrt{9+x^4}\right)(2x)$$

$$f'(2) = \left(\sqrt{9+2^4}\right)(4)$$

$$= 20$$

15. If

$$\int_0^1 \frac{x}{1+x^4}dx \quad \text{is equal to}$$

a) $\frac{1}{4}$

b) $\frac{\pi}{8}$

c) $\frac{\pi}{4}$

d) $\frac{1}{2}$

e) 1

$$= \int_0^1 \frac{x}{1+(x^2)^2} dx$$

(Let $u = x^2$, then $du = 2x\,dx$
$u(0) = 0$, $u(1) = 1$)

$$= \int_0^1 \frac{1}{1+u^2} \frac{1}{2} du = \frac{1}{2} \tan^{-1} u \Big|_0^1$$

$$= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

16) The horizontal asymptotes to the graph of

$$f(x) = \frac{e^x + 5e^{-x}}{e^x + 3e^{-x}} \quad \text{are}$$

a) $y = 1,\ y = \frac{5}{3}$

b) $y = 0$

c) $y = -\frac{1}{2}$ and $y = \frac{3}{5}$

d) $y = 1$

e) This graph does not have any horizontal asymptotes.

$$\lim_{x\to\infty} \frac{e^x+5e^{-x}}{e^x+3e^{-x}} = \lim_{x\to\infty}\left(\frac{e^x+5e^{-x}}{e^x+3e^{-x}}\cdot\frac{e^{-x}}{e^{-x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1+5e^{-2x}}{1+3e^{-2x}}\right) = \frac{1+0}{1+0} = 1$$

Also, $\lim_{x\to-\infty} \frac{e^x+5e^{-x}}{e^x+3e^{-x}} = \lim_{x\to-\infty}\left(\frac{e^x+5e^{-x}}{e^x+3e^{-x}}\cdot\frac{e^x}{e^x}\right)$

$$= \lim_{x\to-\infty} \frac{e^{2x}+5}{e^{2x}+3} = \frac{0+5}{0+3} = \frac{5}{3}$$

17) The vertical asymptotes of the graph of $f(x) = \dfrac{x-4}{(x^2-16)(x^2-9)}$ are

a) $x = -4$, $x = 3$ and $x = -3$

b) $x = -4$, $x = 4$, $x = 3$ and $x = -3$

c) $x = -4$, $x = 4$ and $x = -3$

d) $x = 4$, $x = 3$ and $x = -3$

e) This graph does not have any vertical asymptotes.

Note $f(x) = \dfrac{x-4}{(x-4)(x+4)(x^2-9)} = \dfrac{1}{(x+4)(x^2-9)}$, $x \neq 4$

$\therefore$ Vertical asymptotes are $x=-4$, $x=3$ and $x=-3$.

18) Let $F(x) = \left(1+x^2\right)^x$. $F'(1)$ is equal to

a) 2

b) $1 + 2\ln 2$

c) $\ln 2(1 + 2\ln 2)$

d) $2 + 2\ln 2$

e) 0

Assume $F(x) > 0$.

$\rightarrow \ln F(x) = x \ln(1+x^2)$

$\rightarrow \frac{1}{F(x)} \cdot F'(x) = \ln(1+x^2) + x\left(\frac{2x}{1+x^2}\right)$

$\rightarrow F'(x) = F(x)\left(\ln(1+x^2) + \frac{2x^2}{1+x^2}\right)$

$\rightarrow F'(1) = F(1)\left(\ln 2 + \frac{2}{2}\right)$

$= 2\left(\ln 2 + 1\right) = 2\ln 2 + 2$

19) The area of the region bounded by the curves $y = x^2 - 3x + 2$ and $y = x - 1$ is equal to

a) 1

b) $\frac{2}{3}$

c) $\frac{4}{3}$

d) 3

e) 0

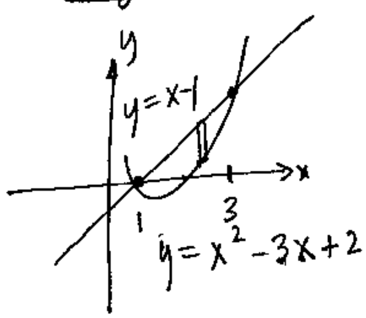
Intersection of curves : $x^2 - 3x + 2 = x - 1$

$\rightarrow x^2 - 4x + 3 = 0$

$\rightarrow (x-3)(x-1) = 0$

$\rightarrow x = 1, 3$

Region



$$\text{Area} = \int_1^3 \left((x-1) - (x^2 - 3x + 2)\right) dx$$

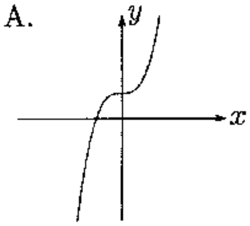
$$= \int_1^3 \left(-x^2 + 4x - 3\right) dx$$

$$= \left(-\frac{x^3}{3} + 2x^2 - 3x\right)\Big|_1^3$$

$$= (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3\right)$$

$$= 4/3$$

20) The graph of the function $f(x) = x^3 - 2x + 1$ looks most like

A.


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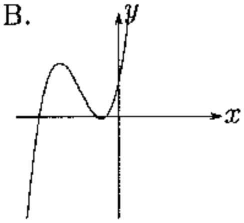

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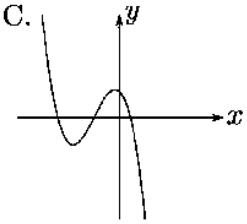

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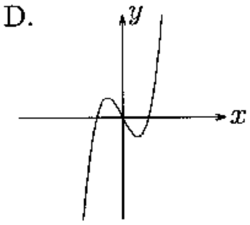
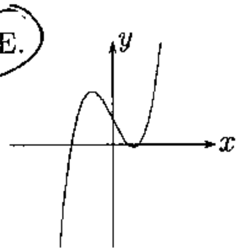


E.


$f'(x) = 3x^2 - 2$

$f''(x) = 6x \Rightarrow$ There is an inflection pt at $x=0$.

$\Rightarrow$ <u>not</u> B or C.

Also $f(0) = 1 \Rightarrow$ <u>not</u> D.

Also $f(1) = 0 \Rightarrow$ <u>not</u> A

$\therefore$ must be E.