SOLUTION KEY

MA 161-161E EXAM 3 SPRING 1999

1. The shortest distance of a point on the graph of $y = \sqrt{x}$ to the point $(4, 0)$ is equal to

A. $2\sqrt{2}$

[✓ B.] $\frac{\sqrt{15}}{2}$

C. 2

D. 3

E. $\frac{5}{2}$

$$D = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$$
$$= \sqrt{x^2 - 7x + 16}$$

$$\frac{dD}{dx} = \frac{2x-7}{2\sqrt{x^2-7x+16}}$$

$$\frac{dD}{dx} = 0 \rightarrow 2x-7=0 \rightarrow x = 7/2$$

$7/2$; x

$2x-7$ $-$ 0 $+$

D dec.   inc $\Rightarrow$ minimum distance at $x = \frac{7}{2}$

$$D\left(\tfrac{7}{2}\right) = \sqrt{\left(\tfrac{7}{2}-4\right)^2 + \left(\sqrt{\tfrac{7}{2}}\right)^2}$$
$$= \sqrt{\frac{1}{4} + \frac{7}{2}} = \sqrt{\frac{15}{4}}$$
$$= \frac{\sqrt{15}}{2}$$

2. $\displaystyle \lim_{x \to -\infty} \frac{6x + 4x^2 + 5x^3}{7x^3 - 3x^2 - 14} =$

A. $\frac{6}{7}$

B. $\frac{4}{7}$

[✓ C.] $\frac{5}{7}$

D. $-\frac{5}{7}$

E. Does not exist

Note: degrees of numerator and denominator are equal (3).

$\therefore$ limit is $\frac{5}{7}$

Alternate solution: divide numerator and denominator by x^3

$$\lim_{x \to -\infty} \frac{\frac{6}{x^2} + \frac{4}{x} + 5}{7 - \frac{3}{x} - \frac{14}{x^2}} = \frac{0+0+5}{7-0-0} = \frac{5}{7}$$

3. All the inflection points of the graph of $f(x) = x^4 - 6x^2$ occur at

A. $x = 0, 3$

B. $x = 1$

C. $x = -1$

D. $x = -1, 1$

E. none

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12 = 0 \Rightarrow x = \pm 1$

$-1 \quad 1 \quad \rightarrow x$

$12(x^2-1) \quad + \quad 0 \quad - \quad 0 \quad +$

$\Rightarrow$ inflection pts occur at $x = 1$ and at $x = -1$.

4. Find the lower sum $L_f(P)$ for $\displaystyle\int_1^4 \frac{1}{x}\,dx$ where $P = \{1, 3, 4\}$.

A. $\frac{5}{12}$

B. $\frac{7}{12}$

C. $\frac{3}{4}$

D. $\frac{11}{12}$

E. $\ln 4$

$1 \quad 3 \quad 4 \quad \rightarrow x$

$\frac{1}{x} = \quad 1 \quad \frac{1}{3} \quad \frac{1}{4}$

$\frac{1}{x}$ is a decreasing function on $[1,4]$, so minimum of $\frac{1}{x}$ occurs at right end pts.

$$L_f(P) = \left(\frac{1}{3}\right)(3-1) + \left(\frac{1}{4}\right)(4-1)$$
$$= \frac{2}{3} + \frac{1}{4} = \frac{11}{12}.$$

5. Let $G(x) = \int_{x^3}^{1} \sin(t^2)dt$. Then $G'(1) =$

A. $3\sin 1$

B. 1

C. $-5\sin 1$

D. -3

E. $-3\sin 1$

$$G(x) = -\int_1^{x^3} \sin(t^2)\,dt$$

$$\Rightarrow G'(x) = -\left(\sin(x^3)^2\right)(3x^2)$$

$$G'(1) = -(\sin 1)(3)$$

6. Find a and b such that

$$\int_2^3 e^{x^2}dx - \int_a^b e^{x^2}dx = \int_4^3 e^{x^2}dx$$

A. $a=2,\ b=3$

B. $a=2,\ b=4$

C. $a=3,\ b=4$

D. $a=4,\ b=3$

E. $a=4,\ b=2$

$$\Rightarrow \int_2^3 e^{x^2}dx - \int_a^b e^{x^2}dx = -\int_3^4 e^{x^2}dx$$

$$\Rightarrow \int_2^3 e^{x^2}dx + \int_3^4 e^{x^2}dx = \int_a^b e^{x^2}dx$$

$$\Rightarrow \int_2^4 e^{x^2}dx = \int_a^b e^{x^2}dx$$

$$\Rightarrow a=2,\ b=4$$

7. For what value of k can the following integral be evaluated directly using a substitution.

$$5\int e^{t^3} t^k dt$$

Let $u = t^3$. Then $du = 3t^2 dt$, and then $t^2 dt = \frac{1}{3} du$. Substituting gives,

$$5\int e^{t^3} t^2 dt = 5\int e^{u} \tfrac{1}{3} du.$$

A. $k = 0$

B. $k = 1$

C. $k = 2$

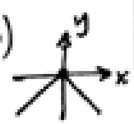
D. $k = 3$

E. $k = 4$

8. True–false: a statement is false unless true in all situations.

a. At a local maximum of a continuous function f we have $f'(x) = 0$.

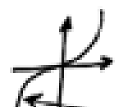
False. let $y = -|x|$. y has a local max at $x=0$ but $y'(0)$ does not exist.



FALSE

b. If $f'(x) > 0$ on (a, b) the graph of f is concave up.

False. Concavity determined by second derivative, not the first. let $f(x) = x^3$. $f'(x) = 3x^2$. $f''(x) = 6x$.



concave down but increasing

FALSE

c. $\int \ln x dx = \frac{1}{x} + C$

$\frac{d}{dx}\left(\frac{1}{x}\right) \neq \ln x$.

False.

d. $\int_0^x \sin t dt$ is increasing for $\frac{\pi}{2} < x < \pi$.

$\frac{d}{dx}\int_0^x \sin t\, dt = \sin x$ and $\sin x > 0$ for $\frac{\pi}{2} < x < \pi$.

TRUE

9. Compute the following integrals

a) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx =$

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\, dx$ and $\frac{1}{\sqrt{x}\, dx} = 2\, du$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \int e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx = \int e^{u}\, 2\, du$$

$$= 2\int e^{u}\, du = 2e^{u} + C = 2e^{\sqrt{x}} + C$$

$$\boxed{2e^{\sqrt{x}} + C}$$

b) $\int \frac{x^3}{1+x^4}\, dx =$

Let $u = 1 + x^4$. Then $du = 4x^3\, dx$ and $x^3 dx = \frac{1}{4} du$.

$$\int \frac{x^3}{1+x^4}\, dx = \int \frac{1}{1+x^4}\, x^3 dx = \int \frac{1}{u} \cdot \frac{1}{4}\, du$$

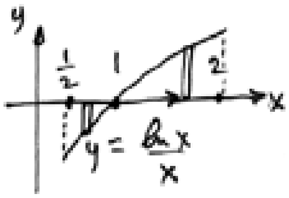
$$= \frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4} \ln|u| + C = \frac{1}{4} \ln(1+x^4) + C$$

$$\boxed{\frac{1}{4} \ln\left(1+x^4\right) + C}$$

10. Give the area between the graph of $f(x) = \dfrac{\ln x}{x}$ and the x axis for $\dfrac{1}{2} \le x \le 2$ as a sum or difference of definite integrals of $\dfrac{\ln x}{x}$ (do not use absolute values). Do not integrate.

$\dfrac{\ln x}{x} < 0$ for $\frac{1}{2} \le x < 1$ and $\dfrac{\ln x}{x} > 0$ for $1 < x \le 2$. $\dfrac{\ln 1}{1} = 0$.

It's not necessary to have an accurate graph of $y = \dfrac{\ln x}{x}$, but you must know whether it's above or below the x-axis.



$$\text{Area} = \int_{\frac{1}{2}}^{1} \left(0 - \frac{\ln x}{x}\right) dx + \int_{1}^{2} \left(\frac{\ln x}{x} - 0\right) dx$$

$$= \boxed{-\int_{\frac{1}{2}}^{1} \frac{\ln x}{x}\, dx + \int_{1}^{2} \frac{\ln x}{x}\, dx}$$

11. In the boxes indicate the substitutions to allow the integrals below to be integrated directly. Do not integrate. (9 points)

a) $\int x\sqrt{x+3}\, dx \quad = \int (u-3)u^{1/2}\, du$

$u = x+3 \to du = dx \quad = \int \left(u^{3/2} - 3u^{1/2}\right) du$

$\to x = u - 3$

$u =$ $x+3$

b) $\int (\ln x)^3 x^{-1} dx \quad = \int u^3\, du$

$u = \ln x \to du = \frac{1}{x}\, dx$

$u =$ $\ln x$

c) $\int \tan^5 x \sec^2 x dx \quad = \int u^5 du$

$u = \tan x \to du = \sec^2 x\, dx$

$u =$ $\tan x$ (or $u = \sec x$)

note: if $u = \sec x$ than $du = \sec x \tan x\, dx$

$$\int \tan^5 x \sec^2 x\, dx = \int \tan^4 x \sec x \sec x \tan x\, dx$$

$$= \int (\sec^2 x - 1)^2 \sec x \sec x \tan x\, dx$$

$$= \int (\sec^5 x - 2\sec^3 x + \sec x) \sec x \tan x\, dx$$

$$= \int (u^5 - 2u^3 + u)^7 du$$

12. Sketch the graph of $f(x) = \ln(e^x + e^{-x})$. Find first and second derivatives and all relative extrema, intervals where graph is concave up and where concave down, and points of inflection. (11 points)

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 0 \rightarrow e^x - e^{-x} = 0 \rightarrow e^x = e^{-x} \rightarrow e^{2x} = e^0$$

$$\rightarrow 2x = \ln e^0 \rightarrow 2x = 0 \rightarrow x = 0.$$

sign of $f'(x)$

		0	
$e^x - e^{-x}$	$-$	0	$+$
f	dec		inc

rel. min

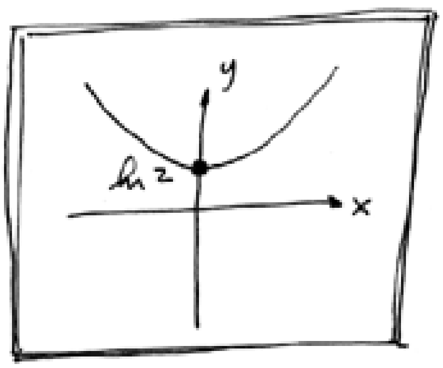
$f(0) = \ln 2$

$$f''(x) = \frac{(e^x - (-e^{-x}))(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$f''(x) > 0$ for all x $\rightarrow$ no inflection pt.



$f'(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$f''(x) = \dfrac{4}{(e^x + e^{-x})^2}$

rel. max. at : none

rel. min. at : $x = 0$

concave up on : all x

concave down on : none

points of inflection : none