**MA161** **FINAL EXAM** **December 15, 1997**

Name: SOLUTION KEY

ID #: ____________________

Recitation Instructor ________________ Time of Recitation ________________

Section #: ____________________

Instructions:

1. Fill in your name, student ID number and division and section numbers on the mark–sense sheet. Also fill in the information requested above.
2. This booklet consists of 14 pages. There are 25 questions, each worth 8 points.
3. Mark your answers on the mark–sense sheet. Please show your working in this booklet.
4. No books, notes or calculators please.
5. When you are finished with the exam, hand this booklet and the mark–sense sheet, in person, to your instructor.
6. Have a nice holiday.

1. If $|2-8x| > 1/2$ then
   - A. $3/16 < x < 5/16$
   - [✓ B.] $x < 3/16$ or $x > 5/16$
   - C. $0 < x < 5/16$
   - D. $5/16 < x < \infty$
   - E. None of the above

$$|2-8x| \geqslant \frac{1}{2}$$

$\rightarrow$ $2-8x \le -\frac{1}{2}$ or $2-8x \geqslant \frac{1}{2}$

| | |
|---|---|
| $2-8x \le -\frac{1}{2}$ | $2-8x \geqslant \frac{1}{2}$ |
| $\rightarrow -8x \le -\frac{5}{2}$ | $\rightarrow -8x \geqslant -\frac{3}{2}$ |
| $\rightarrow x \geqslant \frac{5}{16}$ | $\rightarrow x \le \frac{3}{16}$ |

2. $\lim\limits_{x\to\pi/3} \ln(\ln(2\sin x)) =$
   - [✓ A.] $\ln(\ln 3) - \ln 2$
   - B. $2\ln 3 - \ln 2$
   - C. $\ln 2 + 2\ln 3$
   - D. $\ln(\ln 2) - \frac{1}{3}\ln 2$
   - E. $\ln(\ln 2) - \ln 3$

$$\ln\left(\ln\left(\lim_{x\to\frac{\pi}{3}} 2\sin x\right)\right)$$

$$= \ln\left(\ln\left(2\sin\frac{\pi}{3}\right)\right)$$

$$= \ln\left(\ln\left(2\cdot\frac{\sqrt{3}}{2}\right)\right)$$

$$= \ln(\ln\sqrt{3})$$

$$= \ln(\ln 3^{1/2})$$

$$= \ln\left(\frac{\ln 3}{2}\right)$$

$$= \ln(\ln 3) - \ln(2)$$

3. Consider the tangent to the curve $y = x^3$ at $(2, 8)$. What is the equation of the line that is perpendicular to this tangent and passes through the point $(1, 5)$?

   A. $y - 12x + 7 = 0$

   B. $2y - 5x + 2 = 0$

   C. $y + x - 61 = 0$

   [✓ D.] $12y + x - 61 = 0$

   E. None of the above

Tangent line to curve $y = x^3$ has slope $\frac{dy}{dx} = 3x^2$. At $x=2$, $\frac{dy}{dx} = 12$.

Tangent line is: $y - 8 = 12(x-2)$

$\longrightarrow y = 12x - 16$.

Line perpendicular to tangent line has slope $-\frac{1}{12}$.

Perpendicular line is: $y - 5 = -\frac{1}{12}(x-1)$

$\longrightarrow 12y - 60 = -x + 1 \longrightarrow 12y + x - 61 = 0$.

4. The function $f(x) = \begin{cases} x^2 + 1 & 0 \leq x \leq 3 \\ 10x/3 & 3 < x < \infty \end{cases}$ is

   [✓ A.] continuous for all $x \geq 0$

   B. continuous for all $x \geq 0$ except at $x = 3$

   C. continuous only for $0 \leq x \leq 3$

   D. continuous only for $3 < x < \infty$

   E. None of the above is true

$\lim_{x\to3^-} f(x)$

$= \lim_{x\to3^-} x^2+1 = 10$

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \frac{10x}{3} = 10$.

$\therefore \lim_{x\to3} f(x) = 3 = f(3) \Rightarrow f$ cont. for all $x \geqslant 0$

5. $\lim\limits_{x\to -2} \dfrac{x^2-4}{|x|-2}$

   A. Does not exist

   B. is $-4$

   [✓ C.] is 4

   D. is 0

   E. is $\infty$

$X \to -2$ means $x$ is negative.

$$\therefore \frac{X^2-4}{|X|-2} = \frac{X^2-4}{-X-2} = \frac{(X+2)(X-2)}{-X-2}$$

$$= -(X-2) \text{ provided } X \neq -2.$$

$$\text{Thus } \lim_{X\to -2} \frac{X^2-4}{|X|-2} = \lim_{X\to -2} -(X-2) = 4.$$

6. If $y = \dfrac{\sinh x}{x^2+1}$ then $\dfrac{dy}{dx}$ is

   A. $\dfrac{\cosh x}{2x}$

   B. $\dfrac{x^3\cosh x - x^2\sinh x}{(x^2+1)^2}$

   [✓ C.] $\dfrac{(x^2+1)\cosh x - 2x\sinh x}{(x^2+1)^2}$

   D. $\dfrac{\cosh x - x\sinh x}{(x^2+1)^2}$

   E. None of the above

$$\frac{dy}{dx} = \frac{(\cosh X)(X^2+1) - (\sinh X)(2X)}{(X^2+1)^2}$$

7. Suppose $1 - 2x^2 \leq g(x) \leq -8x + 9$ for $0 \leq x \leq 4$. Then $\lim_{x \to 2} g(x) =$

   A. 2

   [✓ B.] −7

   C. −8

   D. −16

   E. There is not enough information to determine the limit.

Use the squeeze theorem.

$\lim_{x \to 2} 1 - 2x^2 = -7$ and $\lim_{x \to 2} -8 + 9 = -7$.

Therefore $g(x)$ is "squeezed" between $1-2x^2$ and $-8x+9$, so $\lim_{x \to 2} g(x) = -7$.

8. A missile is launched vertically. After $t$ seconds its altitude is $36t \ln(1+t)$ meters above ground. What is its acceleration after 5 seconds?

   [✓ A.] $7\text{m/s}^2$

   B. $9\text{m/s}^2$

   C. $5\text{m/s}^2$

   D. $20\text{m/s}^2$

   E. $25\text{m/s}^2$

let $f(t) = 36t \ln(1+t)$.

Velocity $= f'(t) = 36 \ln(1+t) + 36t\left(\frac{1}{1+t}\right)$

Acceleration $= f''(t) = 36\left(\frac{1}{1+t}\right) + \left[(36)\left(\frac{1}{1+t}\right) + (36t)\frac{-1}{(1+t)^2}\right]$

and $f''(5) = \frac{36}{6} + \left[36\left(\frac{1}{6}\right) + (36 \cdot 5)\left(\frac{-1}{6^2}\right)\right]$

$= 6 + [6 - 5] = 7$.

9. The slope of the tangent line to the curve $x^3 + y^3 + 2y = 4$ at the point $(1, 1)$ is

A. 1

B. $-2/5$

[✓ C. $-3/5$]

D. $3/2$

E. 2

Differentiate implicitly with respect to x :

$$3x^2 + 3y^2 \frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

$$(x,y) = (1,1) \rightarrow 3 + 3\frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

$$\rightarrow \frac{dy}{dx} = \frac{-3}{5}$$

10. Two sides of an isosceles triangle are 3 inches long, and the angle between them is increasing at the rate 1 rad/min. At the moment when the third side is also 3 inches long, at what rate is this side increasing?

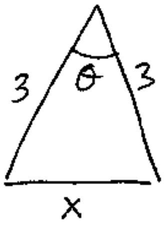
[✓ A. $\frac{3\sqrt{3}}{2}$ in/min]

B. 1 in/min

C. $\frac{\sqrt{3}}{2}$ in/min

D. $\frac{1}{2}$ in/min

E. $\frac{\sqrt{3}}{3}$ in/min



Know: $\frac{d\theta}{dt} = 1 \ \frac{\text{rad}}{\text{min}}$

Want: $\frac{dx}{dt}$ at time when $x = 3$

Law of Cosines gives a relationship between x and θ: $x^2 = 3^2 + 3^2 - 2(3)(3)\cos\theta$.

Differentiate w.r.t. t $\Rightarrow 2x\frac{dx}{dt} = -18(-\sin\theta)\frac{d\theta}{dt}$.

Note: $x = 3 \Rightarrow \theta = \frac{\pi}{3}$. Substituting we get

$$2(3)\frac{dx}{dt} = \left(18 \sin\frac{\pi}{3}\right)(1) \Longrightarrow \frac{dx}{dt} = \frac{18}{6}\left(\frac{\sqrt{3}}{2}\right)(1) = \frac{3\sqrt{3}}{2}$$

11. The sum of two positive angles, $\alpha$ and $\beta$ is $\pi/2$. What is the maximum value of $\sin\alpha + \sin\beta$?

A. 1

B. 3/2

[✓ C.] $\sqrt{2}$

D. $\frac{\sqrt{2}}{2}$

E. There is no maximum

$\alpha + \beta = \pi/2$. Let $S = \sin\alpha + \sin\beta$.

Want: max of $S$.

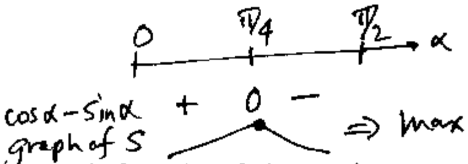
Now, $\beta = \pi/2 - \alpha$, so $S(\alpha) = \sin\alpha + \sin\left(\frac{\pi}{2} - \alpha\right)$

$\therefore S(\alpha) = \sin\alpha + \sin\frac{\pi}{2}\cos\alpha - \cos\frac{\pi}{2}\sin\alpha$

$= \sin\alpha + \cos\alpha$.

$\frac{dS}{d\alpha} = \cos\alpha - \sin\alpha = 0 \rightarrow \sin\alpha = \cos\alpha \Rightarrow \tan\alpha = 1$

$\Rightarrow \alpha = \pi/4$



$\Rightarrow$ max $S$ at $\alpha = \frac{\pi}{4}$

$S\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

12. A function $h$ is continuous and differentiable on $(-\infty, \infty)$. We know $h(0) = 0$ and $h(1) = 2$. Which of the following must be true?

I. On the interval $[0,1]$ $h$ has a maximum.

II. There is an $x$, $0 \le x \le 1$, such that $h'(x) = 0$.

III. There is an $x$, $0 \le x \le 1$, such that $h'(x) = 2$.

A. Only I

B. Only I and II

[✓ C.] Only I and III

D. Only II and III

E. All three

I. TRUE because $h$ is continuous and $[0,1]$ is a closed interval.

II. FALSE By Mean Value Theorem, There is an $x$, $0 \le x \le 1$, such that $h'(x) = \frac{h(1) - h(0)}{1 - 0} = \frac{2-0}{1-0} = 2$

III. TRUE (See explanation for II.)

13. The relative extrema of the function $\ln(e^x + e^{-x})$ are as follows.

A. Relative minimum at 0, relative maxima at $1/e$ and $-1/e$.

B. Relative minimum at $1/e$ and $-1/e$, relative maximum at 0.

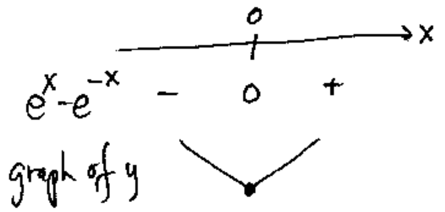
C. There is no relative minimum, there is relative maximum at 0.

[✓ D.] Relative minimum at 0, but there is no relative maximum.

E. There are no relative extrema.

Let $y = \ln(e^x + e^{-x})$. Then $\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{dy}{dx} = 0 \to e^x = e^{-x} \to e^{2x} = 1 \to 2x = 0 \to x = 0$



min at $x = 0$

no max

14. $\lim_{x\to\infty} \frac{x - 1/x + \sin 1/x}{2x + \sqrt{1+x}} =$ $\frac{\infty - 0 - 0}{\infty + \infty} = \frac{\infty}{\infty} = ?$

A. $-1/2$

B. 0

C. $1/3$

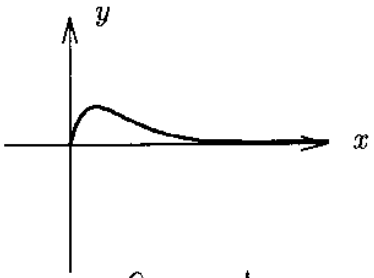
[✓ D.] $1/2$

E. $\infty$

divide numerator and denominator by x:

$$\lim_{x\to\infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x}\sin\frac{1}{x}}{2 + \sqrt{\frac{1}{x^2} + \frac{1}{x}}} = \frac{1}{2}$$

(15.)



. This could be the graph of the function

[✓ A.] $e^{-2x} - e^{-3x},\ x > 0$

B. $1/(1+x),\ x > 0$

C. $x/(1+x),\ x > 0$

D. $1/\ln x,\ x > 0$

E. $xe^{x},\ x > 0$

$\underline{\underline{\text{not}}}$ B. $\lim\limits_{x \to 0} \frac{1}{1+x} = 1.$

$\underline{\underline{\text{not}}}$ D. $0 < x < 1 \Rightarrow \frac{1}{\ln x} < 0.$

$\underline{\underline{\text{not}}}$ E. $\lim\limits_{x \to \infty} xe^{x} = \infty.$

$\underline{\underline{\text{not}}}$ C. $\lim\limits_{x \to \infty} \frac{x}{1+x} = 1.$

$\therefore$ must be A.

16. $\int\limits_{-1}^{2} |x^3| dx =$

[✓ A.] 17/4

B. 15/4

C. 1/2

D. 13/4

E. 11/4

$$= \int_{-1}^{0} |x^3|\, dx + \int_{0}^{2} |x^3|\, dx$$

$$= \int_{-1}^{0} -x^3\, dx + \int_{0}^{2} x^3\, dx$$

$$= \frac{-x^4}{4}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{0}^{2}$$

$$= \left[0 - \left(\frac{-1}{4}\right)\right] + \left[\frac{16}{4} - 0\right] = \frac{1}{4} + \frac{16}{4} = \frac{17}{4}$$

17. If $\int_{-2}^{2} f(x)dx = 0$, which of the following statements must be true?

I. $f(x) = 0$ for all $x$ in $[-2, 2]$

II. $|f(x)| \geq 1$ for some $x$ in $[-2, 2]$

III. $\int_{0}^{-2} f(x)dx = \int_{0}^{2} f(x)dx$

A. All three

B. Only I and III

C. Only I and II

[✓ D.] Only III

E. None

I. FALSE Let $f(x) = x$.

II. FALSE Let $f(x) = 0$.

III. TRUE $\int_{-2}^{2} f(x)\,dx = 0$

$$\rightarrow \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx = 0$$

$$\rightarrow -\int_{-2}^{0} f(x)\,dx = \int_{0}^{2} f(x)\,dx$$

$$\rightarrow \int_{0}^{-2} f(x)\,dx = \int_{0}^{2} f(x)\,dx.$$

18. The area enclosed by the curve $x = y^2$ and the line $y = x - 2$ is

A. 7/2

[✓ B.] 9/2

C. 11/2

D. 13/2

E. 15/2

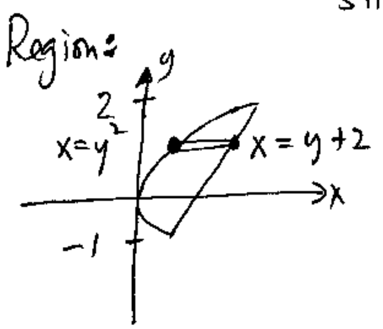
Intersection of curves: $y = y^2 - 2$

$\rightarrow y^2 - y - 2 = 0 \rightarrow (y-2)(y+1) = 0$

$\rightarrow y = -1, 2$

Slice region horizontally

Region:



$$\text{area} = \int_{-1}^{2} (y + 2 - y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}$$

19. $\dfrac{d}{dx}\displaystyle\int_2^{e^x} \frac{dt}{\ln t} =$

$$\left(\frac{1}{\ln e^x}\right)\left(e^x\right) = \frac{e^x}{x}.$$

A. $xe^x$

B. $xe^{-x}$

C. $e^{-x}/x$

[✓ D.] $e^x/x$

E. $1/x$

20. $\dfrac{d}{dx}(2x)^x =$

A. $x(2x)^{x-1}$

B. $(2x)^x \ln 2$

C. $(2x)^x / \ln 2$

D. $(2x)^x \ln(2x)$

[✓ E.] $(2x)^x(1+\ln(2x))$

Let $y=(2x)^x$. Assume $y>0$.

$\rightarrow \ln y = x \ln(2x)$

$\rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = (1)(\ln(2x)) + (x)\left(\dfrac{2}{2x}\right)$

$\rightarrow \dfrac{dy}{dx} = y\left(\ln(2x) + 1\right)$

$= (2x)^x\left(\ln(2x)+1\right),$

21. $\int_0^2 4^x dx =$

A. $8/\ln 2$

B. $8\ln 2$

[✓ C.] $\frac{15}{2\ln 2}$

D. 60

E. 15

$$= \frac{1}{\ln 4} 4^x \Big|_0^2$$

$$= \frac{1}{\ln 4}\left(4^2 - 4^0\right)$$

$$= \frac{1}{\ln 4}\left(16-1\right)$$

$$= \frac{15}{\ln 4} = \frac{15}{2\ln 2}.$$

22. $\int \frac{2x}{\sqrt{1-x^4}}\, dx =$

[✓ A.] $\sin^{-1}(x^2) + C$

B. $\tan^{-1}(x^2) + C$

C. $\ln\sqrt{1-x^4} + C$

D. $\sqrt{1-x^4} + C$

E. $\frac{\sqrt{1-x^4}}{x^2} + C$

$$\int \frac{2x}{\sqrt{1-(x^2)^2}}\, dx$$

$$\left(\text{Let } u = x^2 \to du = 2x\,dx\right)$$

$$= \int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1}(u) + C$$

$$= \sin^{-1}(x^2) + C$$

23. If $f(x) = x^5 + 4x$ then $(f^{-1})'(5)$ is

A. 1

B. 1/4

C. 1/5

[✓ D.] 1/9

E. 1/20

Note: $f(1) = 5$

$$(f^{-1})'(5) = \frac{1}{f'(1)} = \frac{1}{9}$$

$$f'(x) = 5x+4, \quad f'(1) = 9$$

24. $\displaystyle\int_0^{1/3} \frac{dx}{1+9x^2} =$ $\displaystyle = \int_0^{1/3} \frac{1}{1+(3x)^2}\,dx$

A. $\pi/18$

[✓ B.] $\pi/12$

C. $\pi/6$

D. $\pi/3$

E. $\pi/2$

[Let $u = 3x$, Than $du = 3\,dx$
and $u(0) = 0$, $u(1/3) = 1$.]

$$= \int_0^1 \frac{1}{1+u^2}\,\frac{1}{3}\,du = \frac{1}{3}\tan^{-1}u\,\Big|_0^1$$

$$= \frac{1}{3}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$$

25. $\tan(\sin^{-1} x) =$

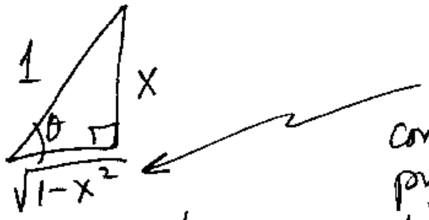
A. $\dfrac{x}{1+x^2}$

B. $\dfrac{1}{1+x^2}$

C. $x\sqrt{1-x^2}$

D. $\dfrac{1}{\sqrt{1-x^2}}$

[✓ E.] $\dfrac{x}{\sqrt{1-x^2}}$



this leg comes from pythagorean theorem.

let $\theta = \sin^{-1} x$

$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$