MA 161–161E **FINAL EXAM** **May 7, 1999**

Name: SOLUTION KEY

ID #: ____________________

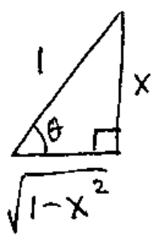
Recitation Instructor ____________ Time of Recitation ____________

Section #: ____________________

Instructions:

1. Fill in your name, student ID number and division and section numbers on the mark–sense sheet. Also fill in the information requested above.
2. This booklet consists of 10 pages. There are 25 questions, each worth 8 points. Each question has exactly one correct answer.
3. Mark your answers on the mark–sense sheet. Please show your work in this booklet.
4. No books, notes or calculators please.
5. When you are finished with the exam, hand this booklet and the mark–sense sheet, in person, to your instructor.

1. $\tan(\sin^{-1}(x)) =$



this leg from pythagorean theorem.

Let $\theta = \sin^{-1} x$

Then $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$

A. $\dfrac{1}{\sqrt{1-x^2}}$

B. $\dfrac{1}{\sqrt{1+x^2}}$

C. $\dfrac{x}{\sqrt{1+x^2}}$

D. $\dfrac{1}{1+x^2}$

E. $\dfrac{x}{\sqrt{1-x^2}}$

2. Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{x-1}$. Find the domain of $f \circ g$.

domain of g is $x \neq 1$.

$(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x-1}\right) = \dfrac{1}{\frac{1}{x-1} - 2}$

$= \dfrac{1}{\frac{1-2(x-1)}{x-1}} = \dfrac{1}{\frac{3-2x}{x-1}} = \dfrac{x-1}{3-2x} \to x \neq \dfrac{3}{2}$.

A. $x \neq 1$ and $x \neq 2$

B. $x \neq 1$ and $x \neq \frac{3}{2}$

C. $x \neq 2$ and $x \neq \frac{3}{2}$

D. $x \neq 1$ and $x \neq 2$ and $x \neq \frac{3}{2}$

E. $x \neq \frac{3}{2}$

3. The graph of $y = x^2 + 4x + 2$ can be drawn by translating the graph of $y = x^2$ so that the vertex (lowest point) of the graph of $y = x^2$ is translated to the point

Complete square $\Rightarrow y = (x^2 + 4x + 4) - 4 + 2$

$\Rightarrow y = (x+2)^2 - 2$

$\Rightarrow y + 2 = (x+2)^2$

$\Rightarrow y - (-2) = (x - (-2))^2 \Rightarrow$ vertex moved to $(-2, -2)$

A. $(2, 2)$

B. $(-2, 2)$

C. $(2, -2)$

D. $(-2, -2)$

E. $(-2, 0)$

4. Suppose $\lim_{x\to 1} f(x) = 3$. Which of the following could be the graph of f?

A. only 1

B. only 1 and 2

C. only 1 and 2 and 3

D. 1 and 2 and 3 and 4

E. only 3

The points on the graphs of 1, 2 and 3 all approach the point (1,3) as $x \to 1$ so for each of 1, 2 and 3 $\lim_{x\to 1} f(x) = 3$.

However, in graph 4, $\lim_{x\to 1^-} f(x) = 3$ but $\lim_{x\to 1^+} f(x) = 2$.

5. Let $f(x) = \dfrac{x^2 - x - 6}{x+2}$. What value of f at $x = -2$ makes f continuous at $x = -2$?

A. 3

B. 2

C. 0

D. -2

E. $\lim_{x\to -2} f(x)$

want: $f(-2) = \lim_{x\to -2} f(x)$.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2-x-6}{x+2}$$

$$= \lim_{x\to -2} \frac{(x+2)(x-3)}{x+2} = \lim_{x\to -2}(x-3) = -5.$$

6. If $f(x) = \dfrac{x^2 - 3x}{2x+1}$, then $f'(1) =$

A. $\frac{1}{9}$

B. $-\frac{2}{3}$

C. $\frac{7}{9}$

D. $-\frac{7}{9}$

E. $\frac{1}{3}$

$$f'(x) = \frac{(2x-3)(2x+1) - (x^2-3x)(2x)}{(2x+1)^2}$$

$$f'(1) = \frac{(-1)(3) - (-2)(2)}{3^2}$$

$$= \frac{1}{9}$$

7. The x-intercept of the line tangent to the graph of $y = 3x^2 - 4x$ at the point $(1, -1)$ is

tangent line has slope $y'(1)$.

$y'(x) = 6x - 4$. $y'(1) = 6 - 4 = 2$

tangent line : $y - (-1) = 2(x-1) \Rightarrow y = 2x - 3$

x-intercept $\rightarrow y = 0 \rightarrow 0 = 2x - 3 \rightarrow x = \frac{3}{2}$

A. $x = -2$
B. $x = \frac{3}{2}$
C. $x = 0$
D. $x = \frac{1}{2}$
E. $x = 2$

8. If $f(x) = \sqrt{\sin(x^3 + 2x)}$, then $f'(1) =$

$$f'(x) = \frac{\cos(x^3+2x)}{2\sqrt{\sin(x^3+2x)}}(3x^2+2)$$

$$f'(1) = \frac{\cos(3)}{2\sqrt{\sin 3}}(5)$$

$$= \frac{5\cos(3)}{2\sqrt{\sin(3)}}$$

A. $\frac{5\cos 3}{2\sqrt{\sin 3}}$
B. $\frac{\cos 3}{2\sqrt{\sin 3}}$
C. $\frac{5\sin 3}{2\sqrt{\cos 3}}$
D. $\frac{\sin 3}{2\sqrt{\cos 3}}$
E. $5\tan 3$

9. Let $y^2 + y = 4x^2 + 2\sin\left(\frac{\pi x}{6}\right) + 1$ define y implicitly as a function of x near $(1, 2)$. Then at $x = 1$, $\frac{dy}{dx} =$

Differentiating with respect to x

$$\Rightarrow 2y\frac{dy}{dx} + \frac{dy}{dx} = 8x + \left(2\cos\left(\frac{\pi x}{6}\right)\right)\left(\frac{\pi}{6}\right) + 0$$

$$(x,y) = (1,2) \Rightarrow 4\frac{dy}{dx} + \frac{dy}{dx} = 8 + \frac{\pi}{3}\cos\frac{\pi}{6}$$

$$\Rightarrow 5\frac{dy}{dx} = 8 + \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{5}\left(8 + \frac{\pi\sqrt{3}}{6}\right)$$

A. $\frac{1}{5}\left(\frac{\pi}{6}\right)$
B. $\frac{1}{3}\left(8 + \frac{\sqrt{3}}{2}\right)$
C. $8 + \pi\frac{\sqrt{3}}{2}$
D. $\frac{1}{5}\left(8 + \frac{\pi\sqrt{3}}{6}\right)$
E. $\frac{1}{3}\left(8 + \frac{\pi}{6}\right)$

10. A substance decays so that $P'(t) = kP(t)$. One third of the substance is lost in 2 hours. How many hours (from the starting time) will it take until only half the substance remains?

$P'(t) = k\,P(t) \Rightarrow P(t) = P(0)\,e^{kt}$

Want to solve $P(t) = \frac{1}{2}P(0)$ for t.

$P(0)\,e^{kt} = \frac{1}{2}P(0) \longrightarrow e^{kt} = \frac{1}{2}$

$\longrightarrow kt = \ln\frac{1}{2} \longrightarrow t = \frac{1}{k}\ln\frac{1}{2}$ (*)

Find k: $P(2) = \frac{2}{3}P(0)$, that is, $\frac{2}{3}P(0)$ remains after 2 hours.

$\Rightarrow P(0)\,e^{2k} = \frac{2}{3}P(0) \rightarrow e^{2k} = \frac{2}{3}$

$\Rightarrow 2k = \ln\frac{2}{3} \rightarrow k = \frac{1}{2}\ln\frac{2}{3}$

$\therefore$ (*) $t = \dfrac{1}{\frac{1}{2}\ln\frac{2}{3}} \cdot \ln\frac{1}{2} = 2\,\dfrac{\ln(1/2)}{\ln(2/3)} = 2\,\dfrac{-\ln 2}{-\ln(3/2)}$

A. $2\dfrac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{2}\right)}$ hours

B. $2\dfrac{\ln 2}{\ln\left(\frac{3}{2}\right)}$ hours

C. $2\dfrac{\ln 2}{\ln 3}$ hours

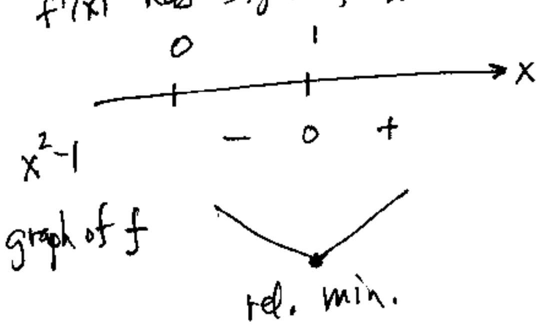
D. $\dfrac{\ln 2}{\ln\left(\frac{3}{2}\right)}$ hours

E. $\dfrac{\ln(2)}{\ln(3)}$ hours

11. The function $f(x) = \frac{1}{2}x^2 - \ln x$

$f'(x) = x - \frac{1}{x} = \dfrac{x^2-1}{x}$.

critical numbers are $x = 0, \pm 1$.
$f'(x)$ has sign of x^2-1.



A. is increasing on $(0,\infty)$

B. has a local minimum at $x = 0$

C. has a local minimum at $x = 1$

D. has a local minimum at $x = \pm 1$

E. has an inflection point at $x = 1$

Note: $f(x)$ defined for $x > 0$

12. Shown below is the graph of $f'(x)$. Then the graph of f is concave down on the interval(s).

A. $[-2.0]$ and $[0, 2]$

B. $[-2, -1]$ and $[0, 1]$

C. $[-2, -1]$

D. $[-1, 0]$ and $[1, 2]$

E. $(-2, 0)$ and $(0, 2]$

f is concave down when f' is decreasing.

f' is decreasing for $-1 \le x \le 0$ and for $1 \le x \le 2$.

Note: f concave down $\Rightarrow f''(x) < 0$

$\Rightarrow f'$ decreasing $\Rightarrow -1 \le x \le 0$ or $1 \le x \le 2$.

13. Let a box with no top have height h and a square base with each side of length x. If the box has volume V then the surface area of the box is

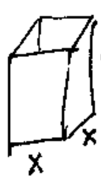
A. x^2h

B. $\pi x^2 h$

C. $x^2 + \frac{4V}{x}$

D. $2x^2 + 4V$

E. $x^2 + 2xh$



volume $V = x^2 y \longrightarrow y = \dfrac{V}{x^2}$

Surface area

$= $ area of bottom $+$ area of sides

$= x^2 + 4xy$

$= x^2 + 4x\left(\dfrac{V}{x^2}\right) = x^2 + \dfrac{4V}{x}$

14. The function $f(x) = \sin x - \dfrac{\sqrt{3}}{2}x$

A. has a local minimum at $x = \frac{\pi}{6}$

B. is always decreasing for $x > 0$

C. has a local minimum at $x = \frac{\pi}{3}$

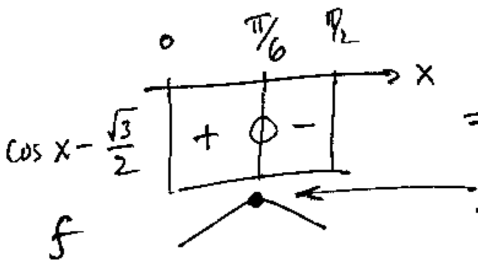
D. has a local maximum at $x = \frac{\pi}{3}$

E. has a local maximum at $x = \frac{\pi}{6}$

$f'(x) = \cos x - \dfrac{\sqrt{3}}{2}$

$= 0 \Rightarrow \cos x = \dfrac{\sqrt{3}}{2}$

$\Rightarrow x = \dfrac{\pi}{6}$ (and other x's)



$\Rightarrow f$ has a relative max at $x = \dfrac{\pi}{6}$

15. The graph shown below is most similar to that of the function

A. $y = \dfrac{1-x}{x^2(x-2)}$

B. $y = \dfrac{x-1}{x^2(x-2)}$

C. $y = \dfrac{x-1}{x(x-2)}$

D. $y = \dfrac{x^2}{(x-1)(x-2)}$

E. $y = \dfrac{(1-x)^2}{x^2(x-2)}$

Consider the sign of y.

(Note: D eliminated since D has vertical asymptote at $x=1$.)

$\boxed{A}$

	0	1	2	
$1-x$	+	+	−	−
$x-2$	−	−	−	+
y	−	−	+	−

$\boxed{B}$

	0	1	2	
$x-1$	−	−	+	+
$x-2$	−	−	−	+
y	+	+	−	+

$\boxed{C}$

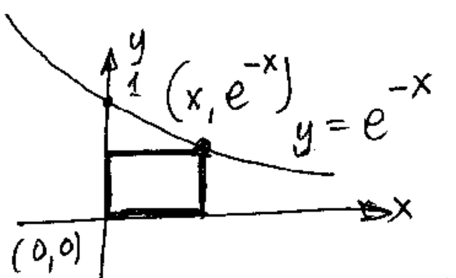
	0	1	2	
$x-1$	−	−	+	+
x	−	+	+	+
$x-2$	−	−	−	+
y	−	+	−	+

$\boxed{E}$

	0	1	2	
$x-2$	−	−	−	+
y	−	−	−	+

Only sign of y in B. matches graph.

16. Let a rectangle be constructed in the first quadrant with one vertex at $(0,0)$ and the opposite vertex at the point (x, e^{-x}). Let $A(x)$ be the area of this rectangle. Then the first derivative $A'(x)$ is



A. $-e^{-x}$

B. $e^{-x}(1-x)$

C. $e^{x}(x-1)$

D. $1 - e^{-x}$

E. xe^{-x}

$$A(x) = xe^{-x}$$

$$A'(x) = (1)(e^{-x}) + (x)(-e^{-x})$$

$$= e^{-x}(1-x)$$

17. $\int_{-1}^{1} \frac{d}{dx} \sqrt{1+x^3}\, dx =$

$\sqrt{1+x^3} \Big|_{-1}^{1}$

$= \sqrt{1+1} - \sqrt{1-1}$

$= \sqrt{2} - 0$

A. $\sqrt{2}$
B. $\frac{3}{2\sqrt{2}}$
C. 0
D. $\frac{3}{\sqrt{2}}$
E. $2\sqrt{2}$

18. $\int_0^1 9(x^2+3)^8 x\, dx =$

$u = x^2+3 \rightarrow du = 2x dx, \; u(0) = 3, \; u(1) = 4.$

$\int_0^1 9(x^2+3)^8 x\, dx = \int_3^4 9(u)^8 \frac{1}{2} du$

$= \frac{9}{2} \cdot \frac{1}{9} u^9 \Big|_3^4 = \frac{1}{2}(4^9 - 3^9)$

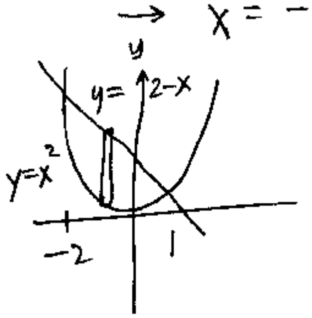
A. $\frac{1}{2} 4^9$
B. $4^9 - 3^9$
C. $\frac{1}{2}(4^9 - 3^9)$
D. 3^9
E. $4^8 - 3^8$

19. The area enclosed by the graphs of $y = x^2$ and $x + y = 2$ is equal to

Intersection of graphs: $x + x^2 = 2$

$\rightarrow x^2 + x - 2 = 0 \rightarrow (x+2)(x-1) = 0$

$\rightarrow x = -2, 1$



Area $= \int_{-2}^{1} (2 - x - (x^2))\, dx$

A. $\int_{-1}^{2} (2 - x - x^2) dx$
B. $\int_{-1}^{2} (x^2 + x - 2) dx$
C. $\int_{-2}^{1} (x^2 + x - 2) dx$
D. $\int_{-2}^{1} (2 - x - x^2) dx$
E. $\int_{-2}^{1} (x^2 + x + 2) dx$

20. If $f(x) = x^3 + 3x - 1$ then the derivative of its inverse at 3, $(f^{-1})'(3)$ is equal to

Note: $f(1) = 1 + 3 - 1 = 3$.

$\therefore (f^{-1})'(3) = \dfrac{1}{f'(1)}$.

$f'(x) = 3x^2 + 3, \quad f'(1) = 6$

Thus, $(f^{-1})'(3) = \dfrac{1}{6}$

A. 6

B. -6

C. $\frac{1}{6}$

D. $-\frac{1}{6}$

E. $\frac{1}{3}$

21. $\dfrac{d}{dx}(x)^{\sin x} =$

Let $y = x^{\sin x}$. Then $\ln y = (\sin x)\ln x$.

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = (\cos x)(\ln x) + (\sin x)\left(\dfrac{1}{x}\right)$

$\Rightarrow \dfrac{dy}{dx} = \left(x^{\sin x}\right)\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

A. $x^{\sin x}\cos x \ln x$

B. $x^{\sin x - 1}$

C. $x^{\sin x}\dfrac{\sin x}{x}$

D. $x^{\sin x}\left\{\cos x \ln x + \dfrac{\sin x}{x}\right\}$

E. $x^{\cos x} + x^{\sin x - 1}$

22. $\displaystyle\int_0^1 5^x dx =$ $\dfrac{1}{\ln 5}\cdot 5^x \Big|_0^1$

$= \dfrac{1}{\ln 5}(5 - 1)$

$= \dfrac{4}{\ln 5}$

A. $\dfrac{4}{\ln 5}$

B. $\dfrac{5}{\ln 5}$

C. 5

D. 1

E. $4\ln 5$

23. $\int_0^{\frac{1}{2}} \frac{1}{4x^2+1}\,dx =$ $\int_0^{\frac{1}{2}} \frac{1}{(2x)^2+1}\,dx$

Let $u = 2x, \rightarrow du = 2dx,\ u(0) = 0,\ u(\frac{1}{2}) = 1.$

$\therefore \int_0^{1/2} \frac{1}{(2x)^2+1}\,dx = \int_0^1 \frac{1}{u^2+1}\,\frac{1}{2}du$

$= \frac{1}{2}\tan^{-1}u\Big|_0^1 = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

A. 1
B. π
C. $\frac{\pi}{2}$
D. $\frac{\pi}{4}$
E. $\frac{\pi}{8}$

24. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^3 x}\,dx =$ $\int_{\pi/4}^{\pi/2} \sin^{-3}x \cos x\,dx$

$= \frac{\sin^{-2}x}{-2}\Big|_{\pi/4}^{\pi/2}$

$= \frac{-1}{2}\left(\frac{1}{\sin^2\frac{\pi}{2}} - \frac{1}{\sin^2\frac{\pi}{4}}\right) = \frac{-1}{2}(1-2) = \frac{1}{2}$

A. 1
B. $-\frac{1}{2}$
C. 0
D. $\frac{1}{2}$
E. 1

25. Let $f(x) = \sin^{-1}(x^2)$ then $f'(x) =$

$f'(x) = \frac{1}{\sqrt{1-(x^2)^2}}(2x)$

$= \frac{2x}{\sqrt{1-x^4}}$

A. $\frac{1}{\sqrt{1-x^2}}$
B. $\frac{1}{\sqrt{1-x^4}}$
C. $\frac{2x}{1+x^4}$
D. $\frac{1}{1+x^4}$
E. $\frac{2x}{\sqrt{1-x^4}}$