MA 162 Final Exam May 2000

NAME Solution Key

STUDENT ID

REC. INSTR. REC. TIME

SECTION NUMBER LECTURER

INSTRUCTIONS:

1. Make sure you have all 12 test pages.
2. Fill in the information requested above and on the mark-sense sheet.
3. Mark your answers on the mark-sense sheet and show work in this booklet.
4. There are 22 problems, worth 9 points each.
5. No books or notes or calculators may be used.
6. **Please, show your work.** It may matter in borderline cases.
7. Have a good summer.

Formulae you may or may not find useful:

$$\mathrm{pr}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||^2}\mathbf{a}$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\,\cos\,x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin\,x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \sec x\,dx = \ln|\sec\,x + \tan\,x| + C$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

$$(1+x)^s = \sum_{n=0}^{\infty}\binom{s}{n}x^n$$

$$\tan 2\theta = \frac{B}{A-C}$$

$$x = X\cos\theta - Y\sin\theta,\quad y = X\sin\theta + Y\cos\theta.$$

1. The vector $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ has length 3 and the same direction as $4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Then $v_1 =$

   A. 3
   B. 2
   C. 1
   D. 0
   E. -2

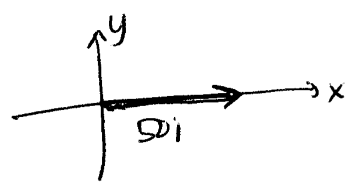
V has same direction as $4i - 2j + 4k$
$\Rightarrow$ there is $k > 0$, such that
$v = k(4i - 2j + 4k)$
$\Rightarrow V_1 = k4, \quad V_2 = k(-2), \quad V_3 = k(4) \qquad (*)$
V has length 3 $\Rightarrow V_1^2 + V_2^2 + V_3^2 = 9 \qquad (**)$
Substituting $(*)$ into $(**)$ gives $(4k)^2 + (-2k)^2 + (4k)^2 = 9$
$\Rightarrow 16k^2 + 4k^2 + 16k^2 = 9 \Rightarrow 36k^2 = 9 \Rightarrow k = \frac{1}{2}$
Therefore $V_1 = k4 = (\frac{1}{2})(4) = 2.$

2. You have to push your broken car 50 meters, by exerting a force of 200 Newtons, at angle $\pi/6$ with respect to the road. How much work will you do on the car?

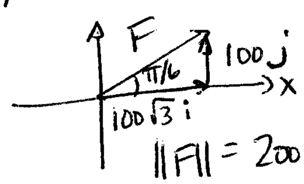
   A. $10,000\sqrt{3}$ Nm
   B. $10,000$ Nm
   C. $5,000\sqrt{3}$ Nm
   D. $5,000$ Nm
   E. none of the above

Work $= \vec{F} \cdot \vec{D}$

$\vec{D} = 50i + 0j$



$\vec{F} = 100\sqrt{3}\, i + 100j$



Therefore Work $= \vec{F} \cdot \vec{D} = 5000\sqrt{3}$

3. A vector perpendicular to both $\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$ is

A. $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
B. $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
C. $4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
D. $2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$
E. $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$(i + 2j + 0k) \times (0i + j + 2k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= i(4-0) - j(2-0) + k(1-0)$$

$$= 4i - 2j + k$$

4. $\lim\limits_{x\to 1^+} \ln x \ln(\ln x) =$

A. 0
B. 1
C. e
D. ∞
E. $-\infty$

$$\lim_{x\to 1^+} (\ln x)(\ln(\ln x)) = 0 \cdot \infty$$

Consider instead, $\lim\limits_{x\to 1^+} \dfrac{\ln(\ln x)}{\frac{1}{\ln x}} = \dfrac{\infty}{\infty}$

Differentiate numerator and denominator:

$$\lim_{x\to 1^+} \frac{\frac{1}{\ln x} \cdot \cancel{\frac{1}{x}}}{\frac{-1}{(\ln x)^2} \cdot \cancel{\frac{1}{x}}} = \lim_{x\to 1^+} -\ln x = 0$$

Therefore, by l'Hôpital's Rule, $\lim\limits_{x\to 1^+} (\ln x)(\ln(\ln x)) = 0$

5. $\int_0^1 (x-1)\, e^{x/2}\, dx =$

A. $2\sqrt{e}$
B. 0
C. $6-4\sqrt{e}$
D. $\sqrt{e}-2$
E. 1

Let $u = x-1$, $dv = e^{x/2}\,dx$

then $du = dx$ and $v = 2e^{x/2}$.

$$\int_0^1 (x-1)e^{x/2}\,dx = (x-1)2e^{x/2}\Big|_0^1 - \int_0^1 2e^{x/2}\,dx$$

$$= \left[(x-1)2e^{x/2} - 4e^{x/2}\right]\Big|_0^1$$

$$= \left(0 - 4e^{1/2}\right) - (-2-4)$$

$$= -4e^{1/2} + 6$$

6. In computing $\int \sin^{-2} x \cos^3 x dx$ which of the following steps will be used?

A. integrate by parts
B. do partial fractions
C. substitute $u = \sin x$
D. substitute $u = \cos x$
E. substitute $u = \sec x$

$$\int \sin^{-2} x \cos^3 x\, dx = \int \frac{\cos^3 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos^2 x \cos x}{\sin^2 x}\,dx$$

$$= \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x}\,dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}\right)\cos x\,dx$$

$$= \int \left(\sin^{-2} x - 1\right)\cos x\,dx$$, let $u = \sin x$.

7. The partial fraction expansion of the function $\frac{x^3+2}{x^2-1}$ will be of form

$$\frac{x^3+2}{x^2-1} = \frac{x^3+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

A. $\frac{A}{x^2-1} + \frac{Bx+C}{x^3+1}$

B. $x + \frac{A}{x-1} + \frac{B}{x+1}$

C. $x^3 + \frac{A}{x-1} + \frac{B}{x+1}$

D. $\frac{A}{x-1} + \frac{B}{x+1}$

E. $\frac{3x}{2} + \frac{Ax+B}{x^2-1}$

8. $\int_0^2 \frac{dx}{(4+x^2)^{3/2}} =$

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$
and $x^2+4 = 4(\tan^2\theta+1) = 4\sec^2\theta$

A. $\frac{\sqrt{2}}{2}$

B. $\frac{\sqrt{2}}{8}$

C. 1

D. $\frac{\pi}{4}$

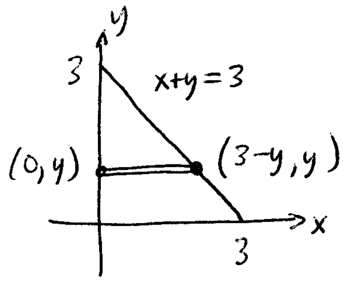
E. $\frac{\pi}{8}$

$$\int_0^2 \frac{dx}{(4+x^2)^{3/2}} = \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{8\sec^3\theta}$$

$$= \int_0^{\pi/4} \frac{1}{4}\frac{1}{\sec\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{4}\cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta \Big|_0^{\pi/4} = \frac{1}{4}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{\sqrt{2}}{8}$$

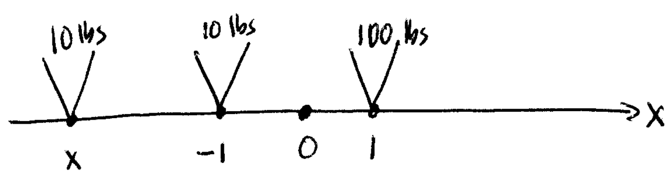
9. The base of a solid is an isosceles right triangle, with legs of length 3. The cross sections perpendicular to one leg are squares. What is the volume of the solid?



A. 6
B. $9\sqrt{2}$
C. 9
D. $\frac{27}{2}$
E. $\frac{27}{\sqrt{2}}$

$$V = \int_0^3 (3-y)^2\,dy = -\frac{(3-y)^3}{3}\Big|_0^3 = 0 - \left(\frac{-3^3}{3}\right) = 9$$

10. Two kids are sitting on opposite sides of a seesaw, both 1 ft from the axis of revolution. One kid weighs 10 lbs, the other 100 lbs. How far from the axis should a third kid, also weighing 10 lbs, sit to achieve equilibrium?



A. 6 ft
B. 8 ft
C. 9 ft
D. 10 ft
E. 11 ft

want: $10x + 10(-1) + 100(1) = 0$

$\longrightarrow 10x = -90$

$\longrightarrow x = -9$

11. $\lim_{k\to\infty} \frac{2\ln k}{\sqrt{k+1}} =$

A. 2
B. $\frac{1}{2}$
C. $\frac{1}{4}$
D. 0
E. 1

$$\lim_{k\to\infty} \frac{2\ln k}{\sqrt{k+1}} = \frac{\infty}{\infty}$$

consider : $\lim_{k\to\infty} \frac{2\left(\frac{1}{k}\right)}{\frac{1}{2\sqrt{k+1}}}$

$$= \lim_{k\to\infty} \frac{4\sqrt{k+1}}{k} = \lim_{k\to\infty} \frac{4\sqrt{k+1}}{\sqrt{k^2}}$$

$$= \lim_{k\to\infty} 4\sqrt{\frac{k+1}{k^2}} = 4\cdot 0 = 0$$

Therefor, by l'Hôpital's Rule, $\lim_{k\to\infty} \frac{2\ln k}{\sqrt{k+1}} = 0$

12. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2} 3^{1-n} =$

A. $\frac{9}{8}$
B. $\frac{3}{4}$
C. $\frac{9}{4}$
D. $\frac{3}{2}$
E. 3

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)(3)\left(\frac{1}{3^n}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)\left(\frac{-1}{3}\right)^n$$ convergent geometric series, $r = -\frac{1}{3}$

$$= \left(\frac{3}{2}\right)\left(\frac{\left(-\frac{1}{3}\right)^0}{1-\left(-\frac{1}{3}\right)}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{\frac{4}{3}}\right) = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right) = \frac{9}{8}$$

13. Which of the following statements is/are true?

I. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges;

II. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$;

III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

A. Only I
B. Only II and III
C. Only III
D. Only II
E. All three are true.

I. False. $\sum^{\infty} \frac{1}{n^{1/2}}$ is a p-series that diverges.

II True.

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = 1$$ and $\sum^{\infty} \frac{1}{2^n}$ converges.

III. True. $\sum^{\infty} \frac{1}{n^2+1}$ converges (compare to $\sum \frac{1}{n^2}$),

so $\sum^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

14. For what positive values of d does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^d+1}}$ converge?

A. $0 < d \leq 1$
B. $1 < d < \infty$
C. $2 < d < \infty$
D. $\frac{1}{2} < d < \infty$
E. $0 < d < \infty$

$$\sum^{\infty} \frac{1}{\sqrt{n^d}} = \sum^{\infty} \frac{1}{n^{d/2}}$$ and

$\frac{d}{2} > 1 \to d > 2$. (p-series).

Thus $\sum^{\infty} \frac{1}{\sqrt{n^d}}$ converges for $d > 2$ and so

$\sum^{\infty} \frac{1}{\sqrt{n^d+1}}$ converges for $d > 2$ by limit comparison

with $\sum^{\infty} \frac{1}{\sqrt{n^d}}$.

15. If $\frac{d}{dx}(\frac{\sin x}{x})$ is written as $\sum_{n=0}^{\infty} a_n x^n$ then a_5 equals

A. $-\frac{1}{6!}$
B. $-\frac{6}{7!}$
C. $-\frac{1}{7!}$
D. $\frac{1}{6\cdot 5!}$
E. $\frac{5}{6!}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = 0 - \frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots$$

Therefore, $a_5 = \frac{-6}{7!}$

16. The Taylor series of the function $\frac{x^2}{1+2x^3}$ is

A. $1 - x^2 + 2x^3 + \ldots$
B. $x^2 - 2x^3 + 2x^4 + \ldots$
C. $x - 2x^3 + 4x^5 + \ldots$
D. $x^2 - 2x^5 + 4x^8 + \ldots$
E. $2x^3 - 4x^5 + 8x^7 + \ldots$

use the geometric series.

$$\frac{x^2}{1+2x^3} = x^2\left(\frac{1}{1-(-2x^3)}\right)$$

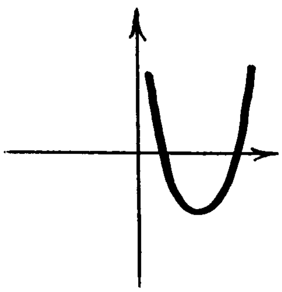
$$= x^2\left(1 + (-2x^3) + (-2x^3)^2 + (-2x^3)^3 + \cdots\right)$$

$$= x^2\left(1 - 2x^3 + 4x^6 - 8x^9 + \cdots\right)$$

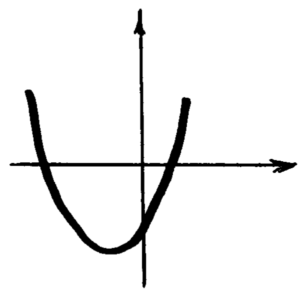
$$= x^2 - 2x^5 + 4x^8 - 8x^{11} + \cdots, \quad |2x^3| < 1$$

17. Which of the following curves is parametrized by $x = 2 - t^2$, $y = t - 1$?

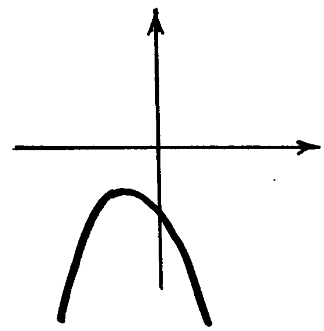
A.



B.



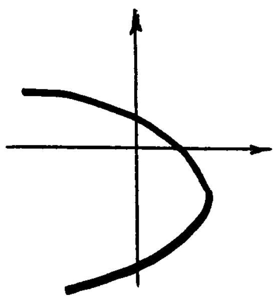
C.

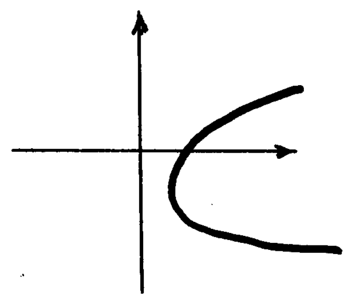


D.

E.

Elimate parameter $\Rightarrow$ $x = 2 - (y+1)^2 \Rightarrow x - 2 = -(y+1)^2$,





which is a parabola opening to left with vertex at $(2, -1)$

18. If a particle travels along the path $x = t^2 - 1$, $y = 2t^3 - 5t^2$, what is its velocity at time $t = 2$?

Let $r(t) = (t^2-1)i + (2t^3 - 5t^2)j$.

Then $v(t) = r'(t)$

$= (2t)i + (6t^2 - 10t)j$.

and $v(2) = 4i + 4j$.

$|v(2)| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

A. $5\sqrt{3}$
B. 8
C. $4\sqrt{2}$
D. 4
E. $2\sqrt{6}$

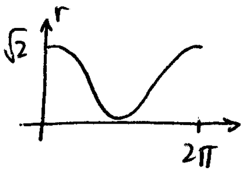
19. Find the area of the region surrounded by the curve $r = \sqrt{1+\cos\theta}$.

A. $\frac{\pi}{2}$
B. π
C. 4π
D. $\frac{\pi}{2}+1$
E. $\pi+2$

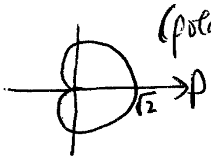
Note: $1+\cos\theta \geq 0$ for all θ.

Therefore r defined for all theta.

Variation of r with θ →



(rectangular coordinate graph of $r=\sqrt{1+\cos\theta}$)



$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)\, d\theta$$

$$= \frac{1}{2}(\theta+\sin\theta)\Big|_0^{2\pi} = \frac{1}{2}(2\pi+0) - \frac{1}{2}(0+0) = \pi$$

20. The equation $r = 2\cos\theta - 4\sin\theta$ describes a circle. What is its center?

A. $(-1, 2)$
B. $(\frac{1}{2}, -1)$
C. $(2, 4)$
D. $(2, -4)$
E. $(1, -2)$

multiply both sides of equation by r ⇒

$$r^2 = 2r\cos\theta - 4r\sin\theta$$

$$\to x^2+y^2 = 2x - 4y$$

$$\to x^2-2x \quad + y^2+4y \quad = 0$$

$$\to x^2-2x+1 \quad + y^2+4y+4 = 5 \quad \text{(complete square)}$$

$$\to (x-1)^2 + (y+2) = 5$$

$\to$ center at $(1, -2)$

21. Find the foci of the ellipse $x^2 - 4x + 4y^2 - 8y + 4 = 0$.

A. $(2, 1 \pm \sqrt{5})$
B. $(-2, \pm\sqrt{3}, -1)$
C. $(\pm\sqrt{3}, 0)$
D. $(0, \pm\sqrt{3})$
E. $(2 \pm \sqrt{3}, 1)$

$x^2 - 4x + 4 \quad + 4(y^2 - 2y + 1) = -4 + 4 + 4$

(completing the squares above)

$\Rightarrow (x-2)^2 + 4(y-1)^2 = 4$

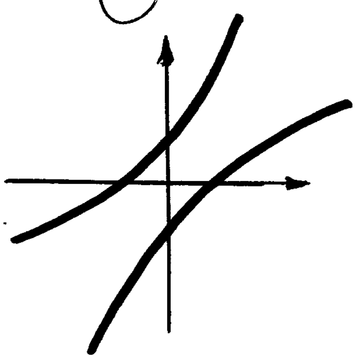
$\Rightarrow \frac{(x-2)^2}{4} + \frac{(y-1)^2}{1} = 1$

$c = \sqrt{4-1} = \sqrt{3}$

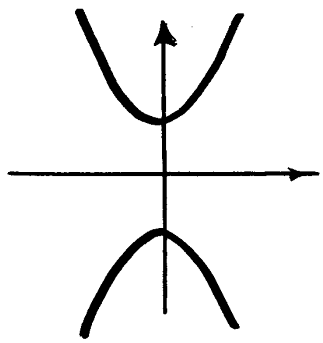
foci at $(2 \pm \sqrt{3}, 1)$

22. Which conic section is described by the equation $x^2 - 4xy + y^2 = 1$

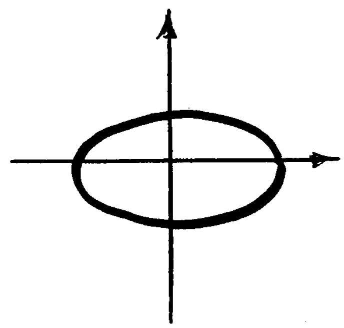
A.



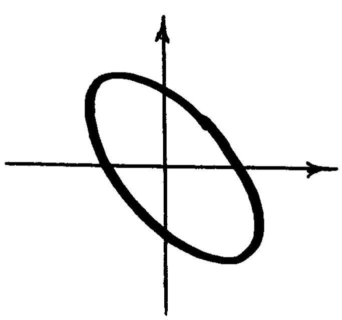
B.



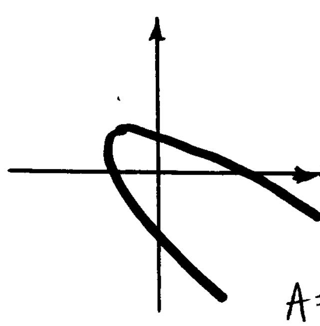
C.



D.



E.



Discriminant is :

$B^2 - 4AC = (-4)^2 - 4(1)(1)$
$= 12 > 0$

Therefore conic is a hyperbola.

$A = C \Rightarrow$ angle of rotation is $\pi/4$.