TEST #1 MA 262 04/01 Oct. 3, 2001

Name Key

(10) 1) Solve: $\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}, \qquad x > 0.$

$p(x) = \frac{2}{x} \qquad \int p(x)dx = 2\int \frac{dx}{x} = 2\ln|x| = \ln x^2$

$\therefore e^{\int p(x)\,dx} = e^{\ln x^2} = x^2$

Mult. thru by x^2 to get

$x^2 \frac{dy}{dx} + 2xy = \sin x$

$\frac{d(x^2 y)}{dx} = \sin x$

$\therefore x^2 y = \int \sin x\,dx + C = -\cos x + c$

$\therefore y = -\frac{\cos x}{x^2} + \frac{c}{x^2}$

$$\boxed{y = \frac{1}{x^2}\left[-\cos x + c\right]}$$

(10) 2) Find the general solution of $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$.

Divide numerator & denominator on rt. by x^2 to give

$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$ Let $y = vx$ Then.

$y' = v'x + v$

and D.E. becomes

$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+v^2}{2v}$

$\therefore \frac{2v}{1+v^2}\,dv = \frac{dx}{x}$

$\ln(1+v^2) = \ln x + C$

$1 + v^2 = Cx$

$v^2 = cx - 1$

$\left(\frac{y}{x}\right)^2 = cx - 1$

$\therefore y^2 = cx^3 - x^2$

$$\boxed{y^2 = c x^3 - x^2}$$

3) Find the general solutions of the following:

(10 pts) (a) $y'' + y' - 6y = 0$.

Auxil. Eqn:

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$y_1 = e^{-3x}$ $y_2 = e^{2x}$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-3x}}$$

(10 pts) (b) $y'' + 12y' + 36y = 0$.

Aux. Eqn

$r^2 + 12r + 36 = 0$

$(r+6)^2$ - double root

$y_1 = e^{-6x}$ $y_2 = xe^{-6x}$

$$\boxed{y = e^{-6x}(c_1 x + c_2)}$$

(10 pts) (c) $y'' - 2y' + 3y = 0$.

Aux eqn:

$r^2 - 2r + 3 = 0$

$r_1, r_2 = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{8}\,i}{2}$

$= \frac{2 + 2\sqrt{2}\,i}{2}$

$= 1 + \sqrt{2}\,i$

$$\boxed{y = e^{x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)}$$

$\therefore$

$y_1 = e^{x}\cos\sqrt{2}x$ $y_2 = e^{x}\sin\sqrt{2}x$

(10 pts) 4) What is the form of the trial function that you would use to find the solution of the non homogeneous problem

$$y'' + y' - 6y = 3e^{-x} + \cos 2x + x^3$$

DO NOT solve for the coefficients. Note that the corresponding homogeneous problem is problem (3a).

$$\boxed{y_p = Ae^{-x} + B\cos 2x + C\sin 2x + a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

(15 pts) 5) Given that $y_1 = x$ and $y_2 = xe^x$ are two solutions of the homogeneous problem corresponding to

$$x^2y'' - x(x+2)y' + (x+2)y = x^2 \quad x > 0,$$

find a particular solution of the non homogeneous problem.

Use method of variation of parameters to find $y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 will satisfy

$u_1' y_1 + u_2' y_2 = 0$ Here: $y_1 = x$ $y_2 = xe^x$

$u_1 y_1' + u_2' y_2' = x^2$ $y_1' = 1$ $y_2' = xe^x + e^x = e^x(x+1)$

Hence:

$u_1' x + u_2' x e^x = 0$ (1)

$u_1' + u_2' e^x(x+1) = x^2$ (2)

Since $x>0$, (1) becomes

$u_1' + u_2' e^x = 0$

$\therefore u_1' = -u_2' e^x$ (3)

Subs. into (2) gives

$u_2' x e^x = x^2$

or, since $x>0$, $u_2' = xe^{-x}$ (4)

$\therefore u_2 = \int xe^{-x}dx \underset{\text{PARTS}}{=} -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$

Subs (4) into (3) gives

$u_1' = -x$

$\therefore u_1 = -\frac{x^2}{2}$

$\therefore$

$y_p = \left(-\frac{x^2}{2}\right)x + \left(-e^{-x}(x+1)\right)xe^x$

$= -\frac{x^3}{2} - x^2 - x$

$$\boxed{y_p = -\frac{x^3}{2} - x^2 - x}$$

Solution to Problem 5 using Green's function.

Note that this involves more work than the other method

$$y_p = \int_0^x \frac{\begin{vmatrix} t & te^t \\ x & xe^x \end{vmatrix}}{\begin{vmatrix} t & te^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt \underset{\substack{\text{prop of}\\ \text{det}}}{=} \int_0^x \frac{t\begin{vmatrix} 1 & e^t \\ x & xe^x \end{vmatrix}}{t\begin{vmatrix} 1 & e^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt$$

$$\underset{\text{Prop of det}}{=} x \int_0^x \frac{\begin{vmatrix} 1 & e^t \\ 1 & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt = x\int_0^x \frac{(e^x - e^t)}{te^t}\, t^2\,dt$$

$$= xe^x \int_0^x \underbrace{t}_{u}\, \underbrace{e^{-t}dt}_{dv} - x\int_0^x t\,dt$$

$$= xe^x\left[-xe^{-x} + \int_0^x e^{-t}dt\right] - \frac{x^3}{2}$$

$$= -x^2 - \frac{x^3}{2} + xe^x \int_0^x e^{-t}dt$$

$$= -\frac{x^3}{2} - x^2 - x + xe^x$$

But xe^x is soln of (H) so can take $y_p = -\frac{x^3}{2} - x^2 - x$

(15 pts) 6) Given that $y = x^2$ is a solution of the differential equation,

$$x^2y'' - 4xy' + 6y = 0 \quad x > 0$$

find a second linearly independent solution and then solve the initial value problem $y(1) = 1 \;\; y'(1) = 1$.

Let $y = ux^2$ be the second linearly independent solution.

Then:

$$\begin{array}{r|l} 6 & y = ux^2 \\ -4x & y' = 2xu + x^2u' \\ x^2 & y'' = 2u + 4xu' + x^2u'' \end{array}$$

$\therefore\; x^2y'' - 4xy' + x^2y'' = u(\underbrace{6x^2 - 8x^2 + 2x^2}_{0}) + u'(\underbrace{-4x^3 + 4x^3}_{0}) + x^4u''$

$\therefore$ For $y = ux^2$ to be soln, ~~since x > 0~~

$x^2u'' = 0$ & since $x > 0$

$u'' = 0$

$u = x$

$y = x^3$

$\therefore$ gen soln is:

$y = C_1x^3 + C_2x^2$

$y' = 3C_1x^2 + 2C_2x$

At $x = 1$: $\begin{matrix} C_1 + C_2 = 1 \\ 3C_1 + 2C_2 = 1 \end{matrix} \Rightarrow C_1 = -1 \;\; C_2 = 1$

second solution	solution to IVP
$y = x^3$	$y = -x^3 + 2x^2$

(10 pts) 7) For what values of k, if any, will the following system have

(a) a unique solution

(b) infinitely many solutions

(c) no solution

$$\begin{aligned} x_1 + \;\; x_2 + \;\; x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 4x_2 + 5x_3 &= k. \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 4 & 5 & k \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & k-3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & k-3 \end{bmatrix}$$

unique None

infinitely many $k = 3$

none $k \neq 3$