TEST #1 MA 262 07/01 Oct. 3, 2001

Name Key

(10) 1) Solve: $\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$, $x > 0$.

$p(x) = \frac{2}{x}$ $\int p(x)\,dx = 2\int \frac{dx}{x} = 2\ln x = \ln x^2$

$\therefore e^{\int p(x)dx} = e^{\ln x^2} = x^2$

Mult. through by x^2 to get

$x^2 \frac{dy}{dx} + 2xy = \sin x$

$\frac{d(x^2 y)}{dx} = \sin x$

$\therefore x^2 y = \int \sin x\,dx + C = -\cos x + C$

$y = -\frac{\cos x}{x^2} + \frac{C}{x^2}$

$$\boxed{y = \frac{1}{x^2}\left[-\cos x + C\right]}$$

(10) 2) Find the general solution of $(x^2 + y)dx + (x + e^y)dy = 0$.

D.E. has form $M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 1$ $\therefore$ is exact.

Let $\phi(x,y) = \int (x^2+y)\,dx + h(y)$

$= \frac{x^3}{3} + xy + h(y)$

Require

$\phi_y = x + h'(y) = x + e^y$

$\therefore h'(y) = e^y$

$h(y) = e^y$

Soln is $\frac{x^3}{3} + xy + e^y = C$

$$\boxed{\frac{x^3}{3} + xy + e^y = C}$$

3) Find the general solutions of the following:

(10 pts) (a) $y'' + y' - 6y = 0$.

Auxil. eqn:

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$\therefore\ y_1 = e^{-3x} \qquad y_2 = e^{2x}$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-3x}}$$

(10 pts) (b) $y'' + 12y' + 36y = 0$.

aux. Eqn:

$$r^2 + 12r + 36 = 0$$

$$(r+6)^2 = 0$$

-6 double root

$$\therefore\ y_1 = e^{-6x} \qquad y_2 = xe^{-6x}$$

$$\boxed{y = e^{-6x}(c_1 + c_2 x)}$$

(10 pts) (c) $y'' - 2y' + 3y = 0$.

Aux. Eqn:

$$r^2 - 2r + 3 = 0$$

$$r_1, r_2 = \frac{2 \pm \sqrt{4-12}}{2}$$

$$= \frac{2 \pm \sqrt{8}\, i}{2}$$

$$= 1 + \sqrt{2}\, i$$

$$\boxed{y = e^{x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)}$$

$$\therefore\ y_1 = e^{x}\cos\sqrt{2}x$$

$$y_2 = e^{x}\sin\sqrt{2}x$$

(10 pts) 4) What is the form of the trial function that you would use to find the solution of the non homogeneous problem

$$y'' - 2y' + 3y = e^x \sin \sqrt{2}\, x + x^2$$

DO NOT solve for the coefficients. Note that the corresponding homogeneous problem is problem (3c).

$$\boxed{y_p = xe^x\left(A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x\right) + a_0 + a_1 x + a_2 x^2}$$

(15 pts) 5) Given that $y_1 = x$ and $y_2 = xe^x$ are two solutions of the homogeneous problem corresponding to

$$x^2y'' - x(x+2)y' + (x+2)y = x^2 \quad x > 0,$$

find a particular solution of the non homogeneous problem.

Use Method of Variation of parameters. $y_p = u_1 y_1 + u_2 y_2$ where u_1 & u_2 satisfy

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = x^2$

Here $y_1 = x \quad y_2 = xe^x$

$y_1' = 1 \quad y_2' = xe^x + e^x = e^x(x+1)$

$\therefore$

$u_1' x + u_2' xe^x = 0 \quad (1)$

$u_1' + u_2' e^x(x+1) = x^2 \quad (2)$

Since $x > 0$, (1) becomes

$u_1' + u_2' e^x = 0$

$\therefore \quad u_1' = -u_2' e^x \quad (3)$

Subs. int (2) gives:

$u_2' xe^x = x^2$

or $u_2' = xe^{-x} \quad (4)$

$\therefore u_2 = \int xe^{-x}dx \overset{\text{integration by parts}}{=} -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} = -e^{-x}(1+x) \quad (5)$

Subs (4) into (3) gives:

$u_1' = -x$

$\therefore u_1 = -\frac{x^2}{2}$

$\therefore y_p = \left(-\frac{x^2}{2}\right)x + \left(-e^{-x}(x+1)\right)xe^{x}$

$y_p = -\frac{x^3}{2} - x^2 - x$

$$\boxed{y_p = \frac{x^3}{2} - x^2 - x}$$

Solution to Problem 5 using Green's function.

Note that this involves more work than the other method

$$y_p = \int_0^x \frac{\begin{vmatrix} t & te^t \\ x & xe^x \end{vmatrix}}{\begin{vmatrix} t & te^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt \underset{\substack{\uparrow \\ \text{prop of} \\ \text{det}}}{=} \int_0^x \frac{t\begin{vmatrix} 1 & e^t \\ x & xe^x \end{vmatrix}}{t\begin{vmatrix} 1 & e^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt$$

$$\underset{\substack{\uparrow \\ \text{Prop of det}}}{=} x\int_0^x \frac{\begin{vmatrix} 1 & e^t \\ 1 & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^t \\ 1 & te^t+e^t \end{vmatrix}}\, t^2\,dt = x\int_0^x \frac{(e^x-e^t)}{te^t}\, t^2\,dt$$

$$= xe^x \int_0^x \underbrace{t}_{u}\,\underbrace{e^{-t}dt}_{dv} - x\int_0^x t\,dt$$

$$= xe^x\left[-xe^{-x} + \int_0^x e^{-t}\,dt\right] - \frac{x^3}{2}$$

$$= -x^2 - \frac{x^3}{2} + xe^x\int_0^x e^{-t}\,dt$$

$$= -\frac{x^3}{2} - x^2 - x + xe^x$$

But xe^x is soln of (H) so can take $y_p = -\frac{x^3}{2} - x^2 - x$

(15 pts) 6) Given that $y = x^2$ is a solution of the differential equation,

$$x^2y'' - 4xy' + 6y = 0 \quad x > 0$$

find a second linearly independent solution and then solve the initial value problem $y(1) = 1 \;\; y'(1) = 1$. Let $y = u x^2$ be second lin indep. soln.

Then:

$$6 \,|\; y = ux^2$$
$$-4x \,|\; y' = 2xu + x^2u'$$
$$x^2 \,|\; y'' = 2u + 4xu' + x^2u''$$

$\therefore\; x^2y'' - 4xy' + 6y = u(6x^2 - 8x^2 + 2x^2) + u'(-4x^3 + 4x^3) + x^2u''$

$\therefore$ For $y = ux^2$ to be a soln

$x^2u'' = 0$ + since $x > 0$

$u'' = 0$

$\therefore\; u = x$

$y_2 = x^3$

$\therefore$ gen soln is

$y = c_1x^3 + c_2x^2$

$y' = 3c_1x^2 + 2c_2x$

$\therefore$ at $x=1$

$1 = c_1 + c_2$
$1 = 3c_1 + 2c_2$ $\Rightarrow$ $c_1 = -1 \quad c_2 = 1$

second solution	solution to IVP
$y_2 = x^3$	$y = -x^3 + 2x^2$

(10 pts) 7) For what values of k, if any, will the following system have

(a) a unique solution

(b) infinitely many solutions

(c) no solution

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2\,x_1 + x_2 + \;\; x_3 &= 3 \\ x_2 + 3x_3 &= k \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 3 & k \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & k+1 \end{bmatrix}$$

unique No Value
infinitely many $k = -1$
none $k \neq -1$