TEST #2 MA 262 04/01 Nov. 9, 2001

Name Key

(10) 1) If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}}$$

(10) 2) Determine whether or not the set $\{\vec{x} = (x_1, x_2, x_3) : x_1^2 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$. Give a clear statement justifying your conclusion.

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be in the set

Then $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ and so

$$(x_1 + y_1)^2 + (x_2 + y_2) + (x_3 + y_3) = \underbrace{x_1^2 + x_2 + x_3}_{\substack{= 0 \\ \vec{x} \text{ in set}}} + \underbrace{y_1^2 + y_2 + y_3}_{\substack{= 0 \\ \vec{y} \text{ in set}}} + 2x_1 y_1$$

$\therefore$ If $x_1 \neq 0$ and $y_1 \neq 0$, $2x_1 y_1 \neq 0$ & so $\vec{x} + \vec{y}$ is Not in set

$\therefore$ set is NOT a subspace

Or: ~~$c\vec{x} = cx_3\ cx_2$~~ $c\vec{x} = (cx_1, cx_2, cx_3)$ and

$$(cx_1)^2 + c_2 x_2 + c_3 x_3 = (c^2 - c)\, x_1^2 + c\,[x_1^2 + \underbrace{x_2 + x_3}_{= 0}] = c(c-1)\, x_1^2 \neq 0$$

if $c \neq 1$ and $x_1 \neq 0$. $\therefore$ Not a subspace

3) a) Find a set of vectors that spans the set of vectors of the form $(s+2r, s-r, 2s+3r)$, r, s, real.

(b) Is your spanning set a basis? Why?

(c) Find the Cartesian equation of the subspace spanned by these vectors.

(a) $(s+2r,\ s-r,\ 2s+3r) = s\,(1,1,2) + r\,(2,-1,3)$

(c) $x = s+2r$
$y = s-r$ for a vector (x,y,z) in the span
$z = 2s+3r$

From 1st 2 eqns get $x-y = 3r$

From 2nd + 3rd eqn get: $\cancel{2y+z=}$ $2y - z = -5r$

$\therefore \dfrac{x-y}{3} = \dfrac{z-2y}{5}$

or: $5x + y - 3z = 0$

(5) spanning set $\vec{v}_1 = (1,1,2)$ $\vec{v}_2 = (2,-1,3)$

(5) Basis Yes No Reason: $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other ∴ are lin indep + so are a basis

(5) Cartesian equation $5x + y - 3z = 0$

(10) 4) Using the definitions, determine whether the vectors $(1,2,3)$, $(1,1,-1)$ and $(3,2,0)$ are linearly dependent or independent.

We must determine whether or not $\exists$ scalars c_1, c_2, c_3 not all zero such that

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}1\\1\\-1\end{bmatrix}+c_3\begin{bmatrix}3\\2\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\text{ ; i.e solutions of }\begin{bmatrix}1&1&3\\2&1&2\\3&-1&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Using Gauss elimination to solve:

$$\begin{bmatrix}1&1&3\\2&1&2\\3&-1&0\end{bmatrix}\sim\begin{bmatrix}1&1&3\\0&-1&-4\\0&-4&-9\end{bmatrix}\sim\begin{bmatrix}1&1&3\\0&1&4\\0&0&7\end{bmatrix}\quad\therefore\text{ only soln is }c_1=c_2=c_3=0$$

and vectors are linearly independent

Linearly independent

(10) 5) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(1,1)=(1,2)$ and $T(-1,2)=(1,-1)$. Find $T(1,0)$.

$(1,1)$ and $(-1,2)$ are a bases for $\mathbb{R}^2$. Hence we can express $(1,0)$ as a linear combination of these vectors, uniquely, and so.

$$c_1\binom{1}{1}+c_2\binom{-1}{2}=\binom{1}{0}\quad\wedge\quad\begin{matrix}c_1-c_2=1\\c_1+2c_2=0\end{matrix}\Rightarrow c_1=-2c_2\Rightarrow -3c_2=1$$

$$\therefore c_2=\frac{-1}{3}\qquad\therefore c_1=\frac{2}{3}$$

Thus:

$$T(1,0)=T\left[\frac{2}{3}(1,1)-\frac{1}{3}(-1,2)\right]=\frac{2}{3}T(1,1)-\frac{1}{3}T(-1,2)$$

$$=\frac{2}{3}(1,2)-\frac{1}{3}(1,-1)=\left(\frac{1}{3},\frac{5}{3}\right)$$

$T(1,0) =$ $(1/3, 5/3)$

6) Let T be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ 5 & 2 & -4 & 1 \end{bmatrix}.$$

(5) (a) Find $T(1,1,-1,1)$

$T(1,1,-1,1) = A\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix}$

$\boxed{(3,-2,12)}$

(10) (b) Find $\ker(T)$.

$\ker(T)$ = null space$(A) = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\}$. Solving $A\vec{x} = \vec{0}$ by Gauss-elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ 5 & 2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & -3 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So:

$x_1 + x_2 + x_3 + 2x_4 = 0$

$x_2 + 3x_3 + 3x_4 = 0$

From second eqn get $x_3 = r$, $x_4 = s$

$x_2 = -3r - 3s = -3(r+s)$

From this and 1st eqn get

$x_1 = -x_2 - x_3 - 2x_4 = 3(r+s) - r - 2s = 2r + s$

$\therefore \ker(T) = \{\vec{x} \in \mathbb{R}^4 : \vec{x} = (2r+s, -3(r+s), r, s)$

$= r(2,-3,1,0) + s(1,-3,0,1)$

(5) (c) Find a basis for $\operatorname{rng}(T)$. Will be columns of A corresp to cols with lead ones in REF A i.e. cols 1 and 2

$\boxed{(1,-1,5), (1,0,2)}$

(5) (d) What are the dimensions of $\operatorname{rng}(T)$ and $\ker(T)$.

$\dim \operatorname{rng}(T) = 2$

$\dim \ker(T) = 2$

(10) 7) Let $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Find the eigenvalues of A and give their multiplicities.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & -4 \\ 0 & 0 & 1 & -1-\lambda \end{vmatrix} = (\lambda-3)^2 \begin{vmatrix} 1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda-3)^2 [(\lambda+1)(\lambda-1)+4]$$

$$= (\lambda-3)^2(\lambda^2+3) = (\lambda-3)^2(\lambda+\sqrt{3}i)(\lambda-\sqrt{3}i)$$

$\boxed{\lambda_1 = 3 \text{ mult } 2 \qquad \lambda_2 = \sqrt{3}i \qquad \lambda_3 = -\sqrt{3}i}$

(10) 8) Given that the matrix $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ has an eigenvalue $-1+i$, find all eigenvectors of A.

Eigenvectors corresponding to $-1+i$ satisfy

$$\begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or: } \begin{bmatrix} 2-i & -1 \\ 5 & -(2+i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by Gauss elimination:

$$\begin{bmatrix} 2-i & 1 \\ 5 & -(2+i) \end{bmatrix} \underset{\substack{\uparrow \\ \text{mult row 1} \\ \text{by } 2+i}}{\sim} \begin{bmatrix} 5 & -(2+i) \\ 5 & -(2+i) \end{bmatrix} \sim \begin{bmatrix} 5 & -(2+i) \\ 0 & 0 \end{bmatrix} \text{ or } 5x_1 - (2+i)x_2 = 0$$

$\therefore$ Eigenvectors corresponding to $-1+i$ are of the form $r\left(\frac{2+i}{5}, 1\right)$

Complex eigenvalues occur in conjugate pairs, so second eigenvalue is $-1-i$; its eigenvectors are conjugates of those for $-1+i$, or $r\left(\frac{2-i}{5}, 1\right)$

$\boxed{r\left(\frac{2+i}{5}, 1\right) \qquad r\left(\frac{2-i}{5}, 1\right)}$