TEST #2 MA 262 07/01 Nov. 9, 2001

Name Key

(10) 1) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}}$$

(10) 2) Determine whether or not the set $\{\vec{x} = (x_1, x_2, x_3) : x_1^2 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$. Give a clear statement justifying your conclusion.

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be in the set.

Then $\vec{x}+\vec{y} = (x_1+y_1, x_2+y_2, x_3+y_3)$ and so

$$(x_1+y_1)^2 + (x_2+y_2) + (x_3+y_3) = \underbrace{(x_1^2+x_2+x_3)}_{\substack{=0 \\ \vec{x} \text{ in set}}} + \underbrace{(y_1^2+y_2+y_3)}_{\substack{=0 \\ \vec{y} \text{ in set}}} + 2x_1y_1$$

Hence if $x_1 \neq$ and $y_1 \neq 0$, $\vec{x}+\vec{y}$ is not in set, since then $2x_1y_1 \neq 0$ $\therefore$ set is NOT a subspace

or: if $c \neq 1$ $c\vec{x} = (cx_1, cx_2, cx_3)$ and

$$(cx_1)^2 + (cx_2) + (cx_3) = \cancel{(c^2-1)x_1^2 + \ldots (c^2-\ldots)}$$
$$= (c^2-c)x_1^2 + c\underbrace{[x_1^2+x_2+x_3]}_{=0 \ \vec{x} \in \text{set}} = c(c-1)x_1^2$$

$\therefore$ if $x_1 \neq 0$, $c \neq 1$, $c\vec{x}$ is ~~1~~ not in set and set is NOT a subspace.

3) a) Find a set of vectors that spans the set of vectors of the form $(s+r, s-2r, 2s+r)$, r, s, real.

(b) Is your spanning set a basis? Why?

(c) Find the Cartesian equation of the subspace spanned by these vectors.

$$(s+r,\ s-2r,\ 2s+r) = s\,(1,1,2) + r\,(1,-2,1)$$

$x = s+r$

$y = s-2r$

$z = 2s+r$

From first two equations

$x-y = 3r$

From second two equations

$2y - z = -5r$

$\therefore \dfrac{x-y}{3} = \dfrac{z-2y}{5}$

~~$x\ y\ z = 0$~~

and so:

$5x + y - 3z = 0$

(5)	spanning set $(1,1,2)$ $(1,-2,1)$
(5)	Basis Yes / No Reason: Vectors are linearly independent; they are not multiples of each other.
(5)	Cartesian equation $5x + y - 3z = 0$

(10) 4) Using the definitions, determine whether the vectors $(1,2,3)$, $(1,1,1)$ and $(1,-1,3)$ are linearly dependent or independent.

We must determine whether or not $\exists$ scalars c_1, c_2, c_3 not all zero such that

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}1\\1\\1\end{bmatrix}+c_3\begin{bmatrix}1\\-1\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{i.e. solutions of} \quad \begin{bmatrix}1&1&1\\2&1&-1\\3&1&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Using Gauss elimination to solve:

$$\begin{bmatrix}1&1&1\\2&1&-1\\3&1&3\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&-1&-3\\0&-2&0\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&1&3\\0&0&6\end{bmatrix}$$

$\therefore$ Only solutions are $c_1=c_2=c_3=0$

$\boxed{\text{Vectors are linearly independent}}$

(10) 5) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(1,1)=(1,2)$ and $T(-1,2)=(1,-1)$. Find $T(0,1)$.

$(1,1)$ and $(-1,2)$ are a basis for $\mathbb{R}^2$. Hence $(0,1)$ is a linear combination of these vectors uniquely and so:

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{aligned}c_1-c_2&=0\\c_1+2c_2&=1\end{aligned} \Rightarrow c_1=c_2 \Rightarrow c_2=\tfrac{1}{3} \Rightarrow c_1=\tfrac{1}{3}$$

$$\therefore\ T(0,1)=T\left[\tfrac{1}{3}(1,1)+\tfrac{1}{3}(-1,2)\right]=\tfrac{1}{3}T(1,1)+\tfrac{1}{3}T(-1,2)$$

$$=\tfrac{1}{3}(1,2)+\tfrac{1}{3}(1,-1)=\left(\tfrac{2}{3},\tfrac{1}{3}\right)$$

$\boxed{T(0,1)=\left(\tfrac{2}{3},\tfrac{1}{3}\right)}$

6) Let T be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ 5 & 2 & -4 & 1 \end{bmatrix}.$$

(5) (a) Find $T(1, -1, 1, 1)$

$A \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

$\boxed{(3, 2, 0)}$

(10) (b) Find $\ker(T)$.

$\ker(T) = \text{null space}(A) = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\}$ Solving $A\vec{x} = \vec{0}$ by Gauss elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ 5 & 2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & -3 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_1 + x_2 + x_3 + 2x_4 = 0$ From second eqn get: $x_3 = r$, $x_4 = s$

$x_2 + 3x_3 + 3x_4 = 0$ $\therefore x_2 = -3r - 3s = -3(r+s)$

From 1st eqn get: $x_1 = -x_2 - x_3 - x_4 = 3(r+s) - r - 2s = 2r + s$

$\therefore \ker(T) = \{\vec{x} \in \mathbb{R}: \vec{x} = (2r+s, -3(r+s), r, s)$

$= r(2, -3, 1, 0) + s(1, -3, 0, 1)$

(5) (c) Find a basis for $\operatorname{rng}(T)$. Will be columns of A corresponding to columns with lead ones in REF A $\therefore$ cols 1 & 2.

$\boxed{(1, -1, 5) \quad (1, 0, 2)}$

(5) (d) What are the dimensions of $\operatorname{rng}(T)$ and $\ker(T)$.

$\dim \operatorname{rng}(T) = 2$
$\dim \ker(T) = 2$

(10) 7) Let $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Find the eigenvalues of A and give their multiplicities.

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & -4 \\ 0 & 0 & 1 & -1-\lambda \end{vmatrix} = (\lambda-3)^2 \begin{vmatrix} 1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (\lambda-3)^2 \left[(\lambda+1)(\lambda-1)+4\right] = (\lambda-3)^2(\lambda^2+3)$$

$$= (\lambda-3)^2(\lambda+\sqrt{3}i)(\lambda-\sqrt{3}i)$$

$\boxed{\lambda_1 = 3 \text{ mult } 2 \qquad \lambda_2 = \sqrt{3}\,i \text{ mult } 1 \qquad \lambda_3 = -\sqrt{3}\,i \text{ mult } 1}$

(10) 8) Given that the matrix $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ has an eigenvalue $-1+i$, find <u>all</u> eigenvectors of A.

~~Ei~~ Coords (x_1, x_2) of eigenvectors corresponding to $-1+i$ satisfy

$$\begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2-i & -1 \\ 5 & -(2+i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by Gauss-elimination

$$\begin{bmatrix} 2-i & 1 \\ 5 & -(2+i) \end{bmatrix} \underset{\substack{\text{mult row 1} \\ \text{by } 2+i}}{\sim} \begin{bmatrix} 5 & -(2+i) \\ 5 & -(2+i) \end{bmatrix} = \begin{bmatrix} 5 & -(2+i) \\ 0 & 0 \end{bmatrix} \text{ and so: } 5x_1 - (2+i)x_2 = 0$$

$\therefore$ Eigenvectors corresponding to $-1+i$ are: $r\left(\frac{2+i}{5}, 1\right)$

Complex eigenvalues occur in conjugate pairs, so 2nd eigenvalue is $-1-i$

$\boxed{r\,(2+i, 5) \qquad s\,(2-i, 5)}$

Its eigenvectors are conjugate to those for $-1+i$ and so are $r\left(\frac{2+i}{5}, 1\right)$