Review # 4

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

You should be able to find the largest interval for which a second order linear initial value problem is guaranteed to have a unique solution.

THEOREM: If p, q and g are continuous in an open interval I and t_0 is in I, then the initial value problem $y''+p(t)y'+q(t)y = g(t),\ y(t_0) = y_0,\ y'(t_0) = y_0'$, has a unique solution in all of the interval I.

The Wronskian of two differentiable functions y_1 and y_2 is
$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$

THEOREM: If y_1 and y_2 are two solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ in some interval I and there is some point t_0 in I such that $W(y_1, y_2)(t_0) \neq 0$, then the general solution of the differential equation is $y = c_1y_1(t) + c_2y_2(t)$.

THEOREM: If Y is a (particular) solution of the differential equation (*) $y''+p(x)y'+q(x)y = g(x)$ and y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$, then the general solution of (*) is $y = c_1y_1 + c_2y_2 + Y$. ■

SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

The differential equation $ay'' + by' + cy = 0$ has characteristic equation $ar^2 + br + c = 0$.
(i) If the characteristic equation has real and unequal roots r_1 and r_2, then the differential equation has general solution $y = c_1e^{r_1t} + c_2e^{r_2t}$.
(ii) If the characteristic equation has unequal complex roots $\alpha \pm i\beta$, then the differential equation has general solution $y = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right)$.
(iii) If the characteristic equation has equal real roots $r_1 = r_2$, then the differential equation has general solution $y = c_1e^{r_1t} + c_2te^{r_1t}$.

REDUCTION OF ORDER

If $y_1(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = 0$, the substitution $y(t) = v(t)y_1(t)$ leads to the differential equation $y_1(t)v'' + (2y_1'(t) + p(t)y_1(t))v' = 0$. If $v(t)$ is a solution of the latter differential equation, then $y_2(t) = v(t)y_1(t)$ is a solution of the original differential equation. If v is not a constant, then the equation $y'' + p(t)y' + q(t)y = 0$ has general solution $y = c_1y_1 + c_2y_2$.

METHOD OF UNDETERMINED COEFFICIENTS

Consider the linear differential operator $L[y] = ay'' + by' + cy$, where a, b, and c are constants. Note: "L" stands for "linear". It is NOT the Laplace transform symbol $\mathcal{L}$ used later in the course. Knowledge of the character of $L[y]$ for certain functions y allows us to determine the form of particular solutions of some differential equations of the form $L[y] = g(t)$.

The differential equation $ay'' + by' + cy = \left(a_0t^n + a_1t^{n-1} + \cdots + a_n\right)e^{\alpha t}$ has solution
$Y = t^s\left(A_0t^n + A_1t^{n-1} + \cdots + A_n\right)e^{\alpha t}$, where s is the number of times α is a root of the characteristic equation $ar^2 + br + c = 0$.
(First, write $Y = \left(A_0t^n + A_1t^{n-1} + \cdots + A_n\right)e^{\alpha t}$, then multiply by a factor of t if the proposed particular solution contains a solution of the homogeneous equation.)

The equations $ay'' + by' + cy = \left(a_0t^n + a_1t^{n-1} + \cdots + a_n\right)e^{\alpha t}\cos(\beta t)$ and $ay'' + by' + cy = \left(a_0t^n + a_1t^{n-1} + \cdots + a_n\right)e^{\alpha t}\sin(\beta t)$ have solutions
$Y = t^s\left(\left(A_0t^n + A_1t^{n-1} + \cdots + A_n\right)e^{\alpha t}\cos(\beta t) + \right.$
$\left.\left(B_0t^n + B_1t^{n-1} + \cdots + B_n\right)e^{\alpha t}\sin(\beta t)\right)$,
where s is the number of times $\alpha + i\beta$ is a root of the characteristic equation $ar^2 + br + c = 0$.
(First, write

$$Y = (A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos(\beta t) + \\ (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin(\beta t),$$

then multiply by a factor of t if the proposed particular solution contains a solution of the homogeneous equation.)

Practice Questions

1. (a) $L[y] = y'' - 3y' + 2y$. Evaluate $L[e^t]$, $L[e^{2t}]$, $L[e^{-t}]$.

(b) $L[y] = y'' - 4y' + 4y$. Evaluate $L[e^{2t}]$, $L[te^{2t}]$, $L[t^2e^{2t}]$.

(c) $L[y] = y'' - 4y' + 5y$. Evaluate $L[e^{2t}\cos t]$, $L[e^{2t}\sin t]$, $L[\sin t]$.

2. Suppose that y_0 is a solution of $t^2y'' + ty' + y = t^2$ and $L[y] = t^2y'' + ty' + y$. Evaluate $L[y_0 + t^2 - 2t + 1]$.

3. Find the largest open interval for which the initial value problem $y'' + \frac{1}{t}y' + \frac{1}{t-2}y = \frac{1}{t-3}$, $y(1) = 3$, $y'(1) = 2$, has a solution.

4. (a) Show that $y_1 = t$ and $y_2 = t^{-1}$ are solutions of the differential equation $t^2y'' + ty' - y = 0$.

(b) Evaluate the Wronskian $W(t, t^{-1})(t)$.

(c) Find the solution of the initial value problem $t^2y'' + ty' - y = 0$, $y(1) = 2$, $y'(1) = 4$.

In Problems 5–7 find the general solution of the homogeneous differential equations in (a) and use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equations in (b) and (c).

5. (a) $y'' - 5y' + 6y = 0$

(b) $y'' - 5y' + 6y = t^2$

(c) $y'' - 5y' + 6y = e^{2t} + \cos(3t)$

6. (a) $y'' - 6y' + 9y = 0$

(b) $y'' - 6y' + 9y = te^{3t}$

(c) $y'' - 6y' + 9y = e^t + \cos(3t)$

7. (a) $y'' - 2y' + 10y = 0$

(b) $y'' - 2y' + 10y = e^t + \cos(3t)$

(c) $y'' - 2y' + 10y = e^t \cdot \cos(3t)$

8. Find the general solution of the differential equation $y'' - y' = 4t$.

9. The differential equation $t^2y'' + ty' - y = 0$ has solution $y_1(t) = t$.

(a) Use the method of reduction of order to find a differential equation satisfied by v, where $y(t) = tv(t)$ is a solution of $t^2y'' + ty' - y = 0$.

(b) Solve the differential equation in (a) to find a solution of $t^2y'' + ty' - y = 0$ that is not a constant multiple of y_1.

(c) Find the general solution of the differential equation $t^2y'' + ty' - y = 0$.

Answers

1. (a) $L[e^t] = 0,\ L[e^{2t}] = 0,\ L[e^{-t}] = 6e^{-t}$

(b) $L[e^{2t}] = 0,\ L[te^{2t}] = 0,\ L[t^2e^{2t}] = 2e^{2t}$

(c) $L[e^{2t}\cos t] = 0,\ L[e^{2t}\sin t] = 0,\ L[\sin t] = 4\sin t - 4\cos t$

2. $L[y_0 + t^2 - 2t + 1] = 6t^2 - 4t + 1$

3. $0 < t < 2$

4. (b) $W(t, t^{-1})(t) = -2t^{-1}$ (c)$y = 3t - t^{-1}$

5. (a) $y = C_1e^{2t} + C_2e^{3t}$

(b) $y = At^2 + Bt + C$

(c) $y = Ate^{2t} + B\cos(3t) + C\sin(3t)$

6. (a) $y = C_1e^{3t} + C_2te^{3t}$

(b) $y = t^2(At + B)e^{3t}$

(c) $y = Ae^t + B\cos(3t) + C\sin(3t)$

7. (a) $y = (C_1\cos(3t) + C_2\sin(3t))e^t$

(b) $y = Ae^t + B\cos(3t) + C\sin(3t)$

(c) $y = t(A\cos(3t) + B\sin(3t))e^t$

8. $y = C_1 + C_2e^t - 2t^2 - 4t$

9. (a) $t^3v'' + 3t^2v' = 0$

(b) $y = t^{-1}$ or $y = a_1t^{-1} + a_2t,\ a_1 \neq 0$

(c) $y = C_1t + C_2t^{-1}$