

## VARIATION OF PARAMETERS

If  $y_1$  and  $y_2$  are solutions of  $y'' + py' + qy = 0$  with  $W(y_1, y_2) \neq 0$ , we can find a particular solution of  $y'' + py' + qy = g$  of the form  $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ . The simplifying assumption (1)  $u_1'y_1 + u_2'y_2 = 0$  gives  $y' = u_1y_1' + u_2y_2'$ . Then substitution of  $y, y'$ , and  $y''$  into  $y'' + py' + qy = g$  and simplifying gives the differential equation (2)  $u_1'y_1' + u_2'y_2' = g$ . Equations (1) and (2) can be solved for  $u_1'$  and  $u_2'$ . This gives the following:

**THEOREM:** If  $p, q$ , and  $g$  are continuous on an open interval  $I$ ,  $y_1$  and  $y_2$  are solutions of the homogeneous differential equation  $y'' + p(t)y' + q(t)y = 0$ , and  $W(y_1, y_2) \neq 0$ , then the nonhomogeneous differential equation

$y'' + p(t)y' + q(t)y = g(t)$  has particular solution

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt$$

and general solution  $y = c_1y_1(t) + c_2y_2(t) + Y(t)$ . ■

## HIGHER ORDER LINEAR EQUATIONS

The theory is similar to that for second order linear equations. This includes the interval in which solutions exist, the form of general solutions of homogeneous and nonhomogeneous equations with constant coefficients, and the method of undetermined coefficients.

## APPLICATIONS

Spring–mass system,  $mu'' + \gamma y' + ky = F_0 \cos(\omega t)$ :

The mass of an unforced system  $mu'' + \gamma y' + ky = 0$  does not oscillate if the system is either critically damped,  $\gamma = 2\sqrt{km}$ , or overdamped,  $\gamma > 2\sqrt{km}$ . Otherwise, the system oscillates.

A forced, undamped system becomes unbounded as  $t \rightarrow \infty$  if and only if  $\omega = \sqrt{k/m}$ .

A damped system is always bounded.

Know how to find steady–state solutions.

Know how to interpret initial conditions and graphs of solutions.

Know how to use the formulas  $R \cos \delta = A$ ,  $R \sin \delta = B$ ,  $R = \sqrt{A^2 + B^2}$ ,  $A \cos(\omega_0 t) + B \sin(\omega_0 t) = R \cos(\omega_0 t - \delta)$ .