

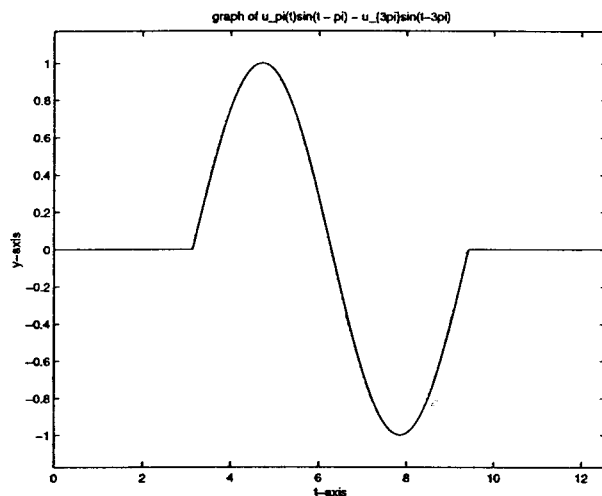
REVIEW # 6

LAPLACE TRANSFORMS

You should be able to express functions that are defined by different formulas on different intervals in terms of the step functions $u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$

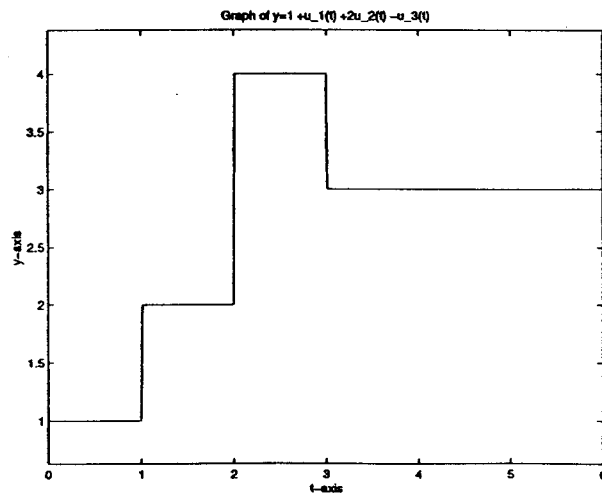
After the vector t has been defined, the MATLAB statement $(t > c)$ gives value zero for $t \leq c$ and value one for $t > c$. The statement $(t > c)$ can then be used for the step function $u_c(t)$ to graph functions that involve step functions.

EXAMPLE To obtain the graph of $y = u_\pi(t) \sin(t - \pi) - u_{3\pi}(t) \sin(t - 3\pi)$:



The jump caused by a step function will appear as a nearly vertical line on your plot.

EXAMPLE To obtain the graph of $y = 1 + u_1(t) + 2u_2(t) - u_3(t)$:



You should be able to use the definition of the Laplace transform as an improper integral to evaluate Laplace transforms.

You should be able to use Table 6.2.1 in the text to evaluate Laplace transforms and inverse Laplace transforms.

- Formula 13 tells us that $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $F(s) = \mathcal{L}\{f(t)\}$. To use Formula 13 to evaluate $\mathcal{L}\{u_c(t)g(t)\}$, we first note that the factor $u_c(t)$ corresponds to the factor e^{-cs} in the Laplace transform. We then note that we need to evaluate the Laplace transform of $f(t)$, where $f(t-c) = g(t)$. We can substitute $t+c$ for t in the formula $f(t-c) = g(t)$ to obtain the formula for $f(t)$ and then evaluate $F(s) = \mathcal{L}\{f(t)\}$.

EXAMPLE Evaluate $\mathcal{L}\{u_1(t)t^2\}$.

SOLUTION We first note that Formula 13 applies with $c = 1$. Also, the function f in Formula 13 must satisfy $f(t-1) = t^2$. Substituting $t+1$ for t in this formula, we obtain $f(t) = (t+1)^2 = t^2 + 2t + 1$. Table 6.2.1 then gives $\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$, so $\mathcal{L}\{u_1(t)t^2\} = e^{-s} \left\{ \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right\}$.

- Formula 13 also says $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where $f(t) = \mathcal{L}^{-1}\{F(s)\}$. To use Formula 13 to evaluate $\mathcal{L}^{-1}\{e^{-cs}F(s)\}$, we first note that the factor e^{-cs} corresponds to the factor $u_c(t)$ in the inverse Laplace transform. We then use Table 6.2.1 to evaluate $f(t) = \mathcal{L}^{-1}\{F(s)\}$. We then substitute $t-c$ for t in the formula for $f(t)$ to obtain $f(t-c)$.

EXAMPLE Evaluate $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s-1)^2}\right\}$.

SOLUTION We first note that Formula 13 applies with $c = 2$. Table 6.2.1 gives $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t$. Substitution of $t-2$ for t in the formula for $f(t)$ gives $f(t-2) = (t-2)e^{t-2}$, so $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s-1)^2}\right\} = u_2(t)(t-2)e^{t-2}$.

PARTIAL FRACTIONS

The partial fraction expansion of a rational function P/Q consists of the sum of a polynomial and terms that have a power of a factor of Q in the denominators. To obtain the partial fraction expansion of P/Q :

- *Check the degrees of the numerator and denominator.*

If $\deg P \geq \deg Q$, divide P by Q to obtain polynomials S and R with $\deg R < \deg Q$ and

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}.$$

If $\deg P < \deg Q$, then $S(x) = 0$, $R(x) = P(x)$, and it is not necessary to divide.

- *Factor the denominator into powers of distinct linear terms and powers of irreducible quadratic terms. This factorization determines the form of the expansion of R/Q .*

A theorem of algebra guarantees that any polynomial with real number coefficients can be factored into powers of linear terms and powers of **irreducible** quadratic terms. The linear factors can be written in the form $x - r$, where r is a zero of the denominator. The quadratic $ax^2 + bx + c$ is irreducible if it has no real zeros. That is, if $b^2 - 4ac < 0$.

- *If $Q(x)$ contains exactly m identical linear factors $x - r$, the expansion of R/Q contains a sum of the form*

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m},$$

where A_1, A_2, \dots, A_m are unknown constants.

- *If $Q(x)$ contains exactly n identical irreducible quadratic factors $ax^2 + bx + c$, the expansion of R/Q contains a sum of the form*

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n},$$

where $B_1, C_1, B_2, C_2, \dots, B_n, C_n$ are unknown constants.

Each distinct factor of Q contributes a term or terms of the type indicated to the expansion of R/Q . In each case, the denominators are powers of the factors, where the powers run from the first power up to the power of the factor in Q . Terms that have powers of linear factors in the denominator have

numerators that are numbers. Terms that have powers of irreducible quadratic factors in the denominator have numerators of the form $(\text{number})x + (\text{number})$.

- *Complete the partial fraction expansion by solving for the values of the unknown constants.*

A theorem of algebra guarantees a unique solution for the values.

Practice Questions

1. $\mathcal{L}\{2 - t^2\} =$

2. $\mathcal{L}\{\sin(2t) - \cos(2t)\} =$

3. $\mathcal{L}\{e^{-t}(1 - \cos(3t))\} =$

4. $\mathcal{L}\{e^{2t}(t^3 + \sin t)\} =$

5. $\mathcal{L}\{u_1(t) + u_2(t)t^2\} =$

6. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2 - t, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad \mathcal{L}\{f(t)\} =$

7. $y'' - 3y' + 2y = 0, y(0) = 1, y'(0) = -1, \quad \mathcal{L}\{y\} =$

8. $y^{(4)} - y = \delta(t), y(0) = 1, y'(0) = -2, y''(0) = 1, y'''(0) = -1, \quad \mathcal{L}\{y\} =$

9. $\mathcal{L}\left\{\int_0^t u_1(t - \tau)e^{t-\tau} \cos(2\tau) d\tau\right\} =$

10. $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 4s + 3}\right\} =$

11. $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} =$

12. $\mathcal{L}^{-1}\left\{\frac{s + 1}{s^2 + 4}\right\} =$

13. $\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 4s + 8}\right\} =$

14. $\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^3}\right\} =$

15. $\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^3}\right)\right\} =$

16. $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2 + 1)}\right\} =$

17. $\mathcal{L}^{-1}\{e^{-s} - e^{-2s}\} =$

18. $\mathcal{L}^{-1}\left\{\frac{1}{s^3} \frac{1}{(s - 1)^3}\right\} = \int_0^t$

19. Use the definition of the Laplace transform as an improper integral to evaluate the Laplace transform of $f(t) = \begin{cases} 0, & 0 \leq t < 2, \\ e^{-t}, & t \geq 2. \end{cases}$

Answers

1. $\frac{2}{s} - \frac{2}{s^3}$

2. $\frac{2}{s^2 + 4} - \frac{s}{s^2 + 4}$

3. $\frac{1}{s + 1} - \frac{s + 1}{(s + 1)^2 + 9}$

4. $\frac{6}{(s - 2)^4} + \frac{1}{(s - 2)^2 + 1}$

5. $\frac{e^{-s}}{s} + e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right)$

6. $\frac{1 - 2e^{-s} + e^{-2s}}{s^2}$

7. $\frac{s - 4}{s^2 - 3s + 2}$

8. $\frac{s^3 - 2s^2 + s}{s^4 - 1}$

9. $\frac{e^{-(s-1)} s}{s - 1} \frac{1}{s^2 + 4}$

10. $\frac{e^{3t} - e^t}{2}$

11. $\frac{t^3}{6}$

12. $\cos(2t) + \frac{1}{2} \sin(2t)$

13. $e^{2t}(\cos(2t) + \sin(2t))$

14. $\frac{t^2 e^t}{2}$

15. $u_2(t) \left((t - 2) - \frac{1}{2}(t - 2)^2 \right)$

16. $u_1(t)(1 - \cos(t - 1))$

17. $\delta(t - 1) - \delta(t - 2)$

18. $\int_0^t \frac{1}{2}(t - \tau)^2 \cdot \frac{1}{2}\tau^2 e^\tau d\tau$

19. $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt = \int_2^\infty e^{-t}e^{-st} dt$
 $= \lim_{b \rightarrow \infty} \int_2^b e^{-(s+1)t} dt = \lim_{b \rightarrow \infty} \frac{e^{-(s+1)b} - e^{-2(s+1)}}{-(s+1)} = \frac{e^{-2(s+1)}}{s+1}$