MA 162 Exam 1 Fall 2000

NAME SOLUTION KEY

STUDENT ID

REC. INSTR. REC. TIME

INSTRUCTIONS:

1. Verify that you have all the pages (there are 5 pages).
2. Fill in your name, your student ID number, and your recitation instructor's name and recitation time above. Write your name, your student ID number and division and section number of your recitation section on your answer sheet, and fill in the corresponding circles.
3. Mark the letter of your response for each question on the mark–sense answer sheet.
4. There are 12 problems worth 8 points each.
5. No books or notes or calculators may be used.
6. Please hand in both your answer sheet and exam to your recitation instructor,

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$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin(2x) = 2 \sin x \cos x$$

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1. What is the distance between the point $(1, 2, -1)$ and the center of the sphere $x^2 + y^2 + z^2 + 2x - 2y + 6z + 7 = 0$.

Find center of sphere by completing squares

$x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 + 6z + 9 = -7 + 1 + 1 + 9$

$\rightarrow (x+1)^2 + (y-1)^2 + (z+3)^2 = 4$

center of sphere at $(-1, 1, -3)$.

Distance between $(-1,1,-3)$ and $(1,2,-1)$

is $\sqrt{(-1-1)^2 + (1-2)^2 + (-3-(-1))^2} = \sqrt{4+1+4} = 3$

A. $\sqrt{11}$
B. 2
C. 3
D. $\sqrt{6}$
E. $\sqrt{7}$

2. Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - t\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + t\mathbf{j} + \mathbf{k}$. For what value of t is $\mathbf{a}$ and $\mathbf{b}$ perpendicular?

a and b perpendicular if and only if $a \cdot b = 0$.

$a \cdot b = 2 + 2t - t = 0$

$\rightarrow 2 + t = 0$

$\rightarrow t = -2$

A. -2
B. -1
C. 0
D. 1
E. 2

3. The parallelogram with sides spanned by $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ has area equal to

area of parallelogram is $\| a \times b \|$.

$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{vmatrix} = i(4+1) - j(2-1) + k(1+2)$

$= 5i - j + 3k$

$|a \times b| = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{35}$

A. 9
B. $\sqrt{35}$
C. 35
D. $\sqrt{27}$
E. $\sqrt{14}$

4. $\lim\limits_{x\to 0} \dfrac{x^3}{1-\cos x} = \dfrac{0}{0}$ Use l'Hopital's Rule.

A. 2
B. 3
C. 6
D. 0
E. ∞

$$\lim_{x\to 0} \frac{3x^2}{\sin x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{6x}{\cos x} = \frac{0}{1} = 0$$

Therefore, $\lim\limits_{x\to 0} \dfrac{x^3}{1-\cos x} = 0$

5. $\lim\limits_{x\to\infty} \left(1+\dfrac{3}{x}\right)^{5x} = 1^{\infty}$

A. e^{15}
B. $e^{\frac{5}{3}}$
C. $e^{\frac{3}{5}}$
D. 15
E. ∞

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{5x} = e^{\lim\limits_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^{5x}} \overset{(*)}{=} e^{15}$$

$$(*) \quad \lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^{5x} = \lim_{x\to\infty} (5x)\ln\left(1+\frac{3}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{5x}} \overset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+3/x}\right)(3)\left(\frac{-1}{x^2}\right)}{\left(\frac{1}{5}\right)\left(\frac{-1}{x^2}\right)} = \frac{(1)(3)}{\frac{1}{5}} = 15$$

6. $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \, dx =$

A. $\dfrac{\pi}{2}$
B. $\dfrac{\pi}{2} - 1$
C. $\dfrac{\pi}{2} + 1$
D. $\dfrac{\pi}{2} + 2$
E. 1

Let $u = x$ and $dv = \sin x\, dx$

Then $du = dx$ and $v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = -x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx$$

$$= (-x\cos x + \sin x)\Big|_0^{\pi/2}$$

$$= (0+1) - (0+0)$$

$$= 1$$

7. $\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x \, dx.$

A. $\frac{1}{5}$

B. 1

C. $\frac{1}{3}$

D. $\frac{2}{15}$

E. $\frac{8}{15}$

$= \int_0^{\pi/4} (\tan^2 x)(\sec^2 x)(\sec^2 x)\, dx$

$= \int_0^{\pi/4} (\tan^2 x)(\tan^2 x + 1)(\sec^2 x)\, dx$

$= \int_0^{\pi/4} (\tan^4 x + \tan^2 x)(\sec^2 x)\, dx$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$

$= \int_0^1 (u^4 + u^2)\, du = \left(\frac{1}{5}u^5 + \frac{1}{3}u^3\right)\Big|_0^1$

$= \frac{1}{5} + \frac{1}{3} = \frac{3+5}{15} = \frac{8}{15}.$

8. $\int_0^{\frac{3}{2}} \frac{x^2}{\sqrt{9-x^2}}\, dx =$

A. $\frac{3}{4}\pi$

B. $9(\frac{\pi}{12} - \frac{1}{8})$

C. $9(\frac{\pi}{12} - \frac{\sqrt{3}}{8})$

D. $\frac{3}{2}\pi$

E. $(\frac{\pi}{6} - \frac{\sqrt{3}}{4})$

Let $x = 3\sin\theta$. Then $9 - x^2 = 9\cos^2\theta$,

and $dx = 3\cos\theta\, d\theta$.

$\int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}}\, dx = \int_0^{\pi/6} \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta\, d\theta$

$= 9\int_0^{\pi/6} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$

$= 9\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/6} = 9\left(\frac{\pi}{12} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right) - 0$

9. In order to compute $\int \frac{1}{(x^2+2x+3)^{\frac{3}{2}}}\, dx$, which of the substitutions would be most effective?

A. $x + 1 = \tan u$

B. $x + \frac{1}{2} = \sqrt{2}\tan u$

C. $x = \sqrt{3}\tan u$

D. $x = \sec u$

E. $x + 1 = \sqrt{2}\tan u$

$x^2 + 2x + 3 = (x^2 + 2x + 1) + 2$

$= (x+1)^2 + 2$

Let $x + 1 = \sqrt{2}\tan u$

(then $(x+1)^2 + 2 = 2\tan^2 u + 2 = 2\sec^2 u$)

10. $\displaystyle\int_2^3 \frac{-x+4}{x^2+x-2}\,dx =$

A. $2\ln 5 - 3\ln 2$

B. $2\ln 5 - 5\ln 2$

C. $2\ln 5$

D. $5\ln 2 - 2\ln 5$

E. $5\ln 2$

$$\frac{-x+4}{x^2+x-2} = \frac{-x+4}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$\to -x+4 = A(x-1) + B(x+2)$$

$x=-2 \to 6 = -3A + 0B \to A = -2$

$x=1 \to 3 = 0A + 3B \to B = 1$

$$\int_2^3 \left(\frac{-2}{x+2} + \frac{1}{x-1}\right)dx = \left(-2\ln|x+2| + \ln|x-1|\right)\Big|_2^3 = (-2\ln 5 + \ln 2) - (-2\ln 4 + \ln 1)$$

$= -2\ln 5 + \ln 2 + 2\ln 2^2$

$= -2\ln 5 + \ln 2 + 4\ln 2$

$= -2\ln 5 + 5\ln 2$

11. Find the form of partial fraction expansion of $\dfrac{3x^2+2}{(x^3-x^2)(x^2+1)}$.

A. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1}$

B. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$

C. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^2+1}$

D. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx}{x^2+1}$

E. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{Dx+E}{x^2+1}$

$$\frac{3x^2+2}{x^2(x-1)(x^2+1)} =$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

12. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx =$

A. ∞

B. $\dfrac{1}{(\ln 2)^2}$

C. $\dfrac{1}{\ln 2}$

D. $\dfrac{1}{2\ln 2}$

E. $\dfrac{-1}{2\ln 2}$

$$= \lim_{c\to\infty} \int_2^c \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{c\to\infty} \left(-\frac{1}{\ln x}\Big|_2^c\right)$$

$$= \lim_{c\to\infty} \left(-\frac{1}{\ln c} + \frac{1}{\ln 2}\right) = 0 + \frac{1}{\ln 2}$$