NAME SOLUTION KEY

STUDENT ID

REC. INSTR. REC. TIME

INSTRUCTIONS:

1. Verify that you have all the pages (there are 5 pages).
2. Fill in your name, your student ID number, and your recitation instructor's name and recitation time above. Write your name, your student ID number and division and section number of your recitation section on your answer sheet, and fill in the corresponding circles.
3. Mark the letter of your response for each question on the mark–sense answer sheet.
4. There are 12 problems worth 8 points each.
5. No books or notes or calculators may be used.

Let R be the region between the graphs of f and g on $[a,b]$. Then the moments of R about x and y axes are

$$M_{y=0} = M_x = \int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)dx$$
$$M_{x=0} = M_y = \int_a^b x(f(x) - g(x))dx.$$

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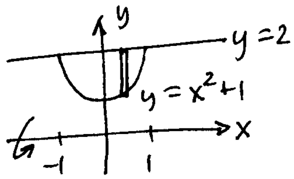
1. Find the volume of the solid generated by revolving about the x axis the region between the graphs of $y = x^2 + 1$ and $y = 2$.



A. $\frac{52}{15}\pi$

B. $\frac{64}{15}\pi$

C. $\frac{25}{6}\pi$

D. 4π

E. $\frac{9}{2}\pi$

$$V = \int_{-1}^{1} \pi\left(2^2 - (x^2+1)^2\right) dx$$
$$= \int_{-1}^{1} \pi\left(4 - (x^4+2x^2+1)\right) dx$$
$$= \int_{-1}^{1} \pi\left(3 - x^4 - 2x^2\right) dx$$
$$= \pi\left(3x - \frac{1}{5}x^5 - \frac{2}{3}x^3\right)\Big|_{-1}^{1} = 2\pi\left(3 - \frac{1}{5} - \frac{2}{3}\right) = \frac{64\pi}{15}$$

2. Suppose $f(x)$ is defined on the interval $[1, 2]$ and $f'(x) = \sqrt{x^6 - 1}$. Find the length of the graph of f.

A. $\frac{15}{4}$

B. 7

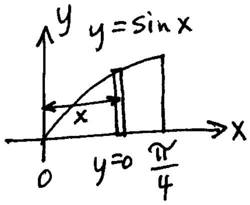
C. 4

D. 15

E. $\frac{63}{7}$

$$L = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$
$$= \int_1^2 \sqrt{1 + (x^6 - 1)}\, dx$$
$$= \int_1^2 x^3\, dx$$
$$= \frac{1}{4}x^4\Big|_1^2 = \frac{1}{4}(16-1) = \frac{15}{4}$$

3. Find the volume of the solid generated by revolving about the y axis the region between $y = \sin x$ and the x axis on the interval $\left[0, \frac{\pi}{4}\right]$.



A. 1

B. $\sqrt{2}\pi\left(1 + \frac{\pi}{4}\right)$

C. 2π

D. $\sqrt{2}\pi\left(1 - \frac{\pi}{4}\right)$

E. π

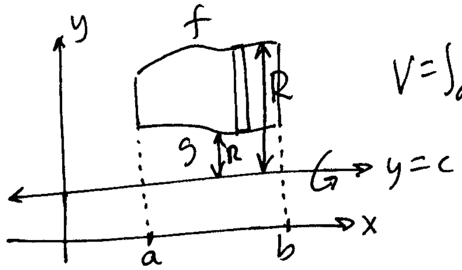
$$V = \int_0^{\pi/4} 2\pi x \sin x\, dx$$

Let $u = 2\pi x$ and $dv = \sin x\, dx$

Then $du = 2\pi\, dx$ and $v = -\cos x$

$$= -2\pi x \cos x\Big|_0^{\pi/4} - \int_0^{\pi/4} -2\pi \cos x\, dx$$
$$= \left(-2\pi x \cos x + 2\pi \sin x\right)\Big|_0^{\pi/4} = -2\pi\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) + 2\pi\left(\frac{\sqrt{2}}{2}\right)$$
$$= \sqrt{2}\pi\left(-\frac{\pi}{4} + 1\right)$$

4. Suppose f and g are continuous on $[a,b]$, and let c be such that $c \le g(x) \le f(x)$ for x in $[a,b]$. The formula for the volume of solid obtained by revolving about the line $y=c$ the region between the graphs of f and g on $[a,b]$ is



$V=\int_a^b \pi(R^2-r^2)\,dx$

$V=\int_a^b \pi\left((f(x)-c)^2-(g(x)-c)^2\right)dx$

A. $\pi\int_a^b (f(x)^2-g(x)^2-c^2)dx$

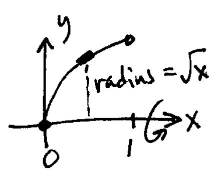
B. $\pi\int_a^b (f(x)^2-g(x)^2)dx$

C. $\pi\int_a^b (f(x)-g(x)-c)^2dx$

D. $\pi\int_a^b ((f(x)-c)^2-g(x)^2)dx$

E. $\pi\int_a^b ((f(x)-c)^2-(g(x)-c)^2)dx$

5. Find the area of the surface generated by revolving about the x axis the graph of $f(x)=\sqrt{x}$ on the interval $[0,1]$.



area $= \int_0^1 2\pi\sqrt{x}\sqrt{1+(f'(x))^2}\,dx$

$= \int_0^1 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = \int_0^1 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx$

$= \int_0^1 \pi\sqrt{4x+1}\,dx = \frac{1}{4}\cdot\frac{2}{3}\pi(4x+1)^{3/2}\Big|_0^1$

$= \frac{\pi}{6}(5^{3/2}-1)$.

A. $\sqrt{8}-1$

B. $2\pi(5^{\frac{3}{2}}-1)$

C. $\frac{4\pi}{3}(\sqrt{8}-1)$

D. $\frac{\pi}{6}(5^{\frac{3}{2}}-1)$

E. $2\pi\sqrt{8}$

6. A force of 5 lbs is needed to hold a spring 6 in. beyond its natural length. How much work is required to stretch the spring an additional 6 in.?

$F=kx \rightarrow 5=k(\frac{1}{2}) \rightarrow k=10 \rightarrow F=10x$.

Work $= \int_{\frac{1}{2}}^1 10x\,dx = 5x^2\Big|_{\frac{1}{2}}^1$

$= 5(1-\frac{1}{4}) = \frac{15}{4}$ ft-lbs.

A. $3\frac{3}{4}$ ft-lb

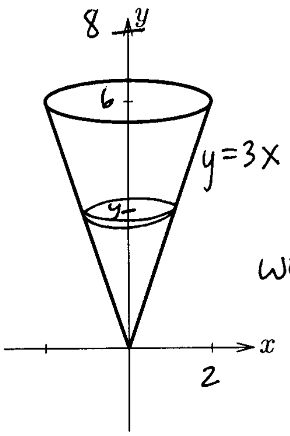
B. $7\frac{1}{2}$ ft-lb

C. 45 ft-lb

D. 30 ft-lb

E. 6 ft-lb

7. A tank is generated by revolving the line $y = 3x$, $0 \le x \le 2$ around the y-axis. If the tank is full of water, how much work is required to pump all the water in the tank to a height 2 feet above the tank? (density of water is 62.5 lbs/ft^3)



$y = 3x \to x = \frac{1}{3}y$

$$\text{work} = \int_0^6 \underbrace{(8-y)}_{\text{distance}} \cdot \underbrace{(62.5)\pi\left(\frac{y}{3}\right)^2}_{\text{force}} dy$$

A. $\int_0^8 (6-y)(62.5)\pi(3y)^2 dy$

B. $\int_0^8 (8-y)(62.5)\pi\left(\frac{y}{3}\right)^2 dy$

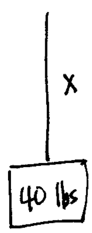
C. $\int_0^6 (8-y)(62.5)\pi(3y)^2 dy$

D. $\int_0^6 (8-y)(62.5)\pi\left(\frac{y}{3}\right)^2 dy$

E. $\int_0^6 (6-y)(62.5)\pi\left(\frac{y}{3}\right)^2 dy$

8. From the top of a crane hangs a 20 foot cable with a 40 pound ball at the end. How much work is required to raise the ball 10 feet? The cable has a linear density of 5 pounds per foot.

cable weighs 5 lbs/foot. Let x = length of cable.



$$\text{work} = \int_{10}^{20} \underbrace{(40+5x)}_{\text{force}} \underbrace{dx}_{\text{distance}}$$

$$= \left(40x + \frac{5}{2}x^2\right)\Big|_{10}^{20} = (800 + 1000) - (400 + 250)$$

$$= 1800 - 650 = 1150$$

A. 1350 ft-lbs

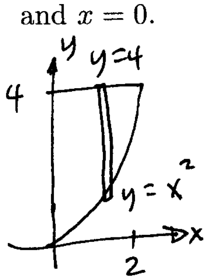
B. 1150 ft-lbs

C. 900 ft-lbs

D. 750 ft-lbs

E. 400 ft-lbs

9. Find the x coordinate of the center of gravity of the region bounded by $y = x^2$, $y = 4$ and $x = 0$.



$$\text{area} = \int_0^2 (4 - x^2)\,dx = \left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= \left(8 - \frac{8}{3}\right) - (0) = \frac{16}{3}$$

$$M_{x=0} = \int_0^2 x(4-x^2)\,dx$$

$$= \int_0^2 (4x - x^3)\,dx = \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2$$

$$= (8-4) - (0) = 4$$

$$\bar{x} = \frac{M_{x=0}}{\text{area}} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4}.$$

A. $\frac{3}{4}$

B. $\frac{3}{16}$

C. $\frac{64}{5}$

D. $\frac{12}{5}$

E. $\frac{16}{3}$

10. Let $p_4(x) = 2 + x - \frac{1}{3}x^2 + \frac{1}{4}x^3 - \frac{1}{5}x^4$ be the fourth Taylor Polynomial for a function $f(x)$. Find $f'''(0)$.

$$\frac{f'''(0)}{3!}x^3 = \frac{1}{4}x^3$$ (Remember the form of terms of Taylor Polyn.)

$$\rightarrow \frac{f'''(0)}{3!} = \frac{1}{4} \rightarrow f'''(0) = \frac{3!}{4} = \frac{6}{4} = \frac{3}{2}$$

A. 0
B. 2
C. $\frac{1}{4}$
D. $\frac{3}{4}$
E. $\frac{3}{2}$

11. Let $p_3(x)$ be the third Taylor Polynomial of $f(x) = \ln(1-x)$. Find $p_3(1)$.

$f(x) = \ln(1-x)$ $\quad f(0) = 0$

$f'(x) = \frac{-1}{1-x}$ $\quad f'(0) = -1$

$f''(x) = \frac{-1}{(1-x)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{-2}{(1-x)^3}$ $\quad f'''(0) = -2$

$P_3(x) = 0 - x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3$

$P_3(1) = 0 - 1 - \frac{1}{2} - \frac{2}{6} = -\frac{11}{6}$

A. 0
B. $\frac{-5}{6}$
C. $\frac{-7}{6}$
D. $\frac{-10}{6}$
E. $\frac{-11}{6}$

12. Rank these limits by size:

a. $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$

b. $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = \lim_{n\to\infty} (2)\frac{n^2}{(n+1)^2} = 2.$

c. $\lim_{n\to\infty} 2^n = \infty$

d. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Therefore $a < d < b < c$

A. $a < b < c < d$
B. $b < a < c < d$
C. $a < d < b < c$
D. $a < b < d < c$
E. $b < a < d < c$