NAME SOLUTION KEY

STUDENT ID

REC. INSTR. REC. TIME

INSTRUCTIONS:

1. Verify that you have all the pages (there are 5 pages).
2. Fill in your name, your student ID number, and your recitation instructor's name and recitation time above. Write your name, your student ID number and division and section number of your recitation section on your answer sheet, and fill in the corresponding circles.
3. Mark the letter of your response for each question on the mark–sense answer sheet.
4. There are 12 problems worth 8 points each.
5. No books or notes or calculators may be used.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$(1+x)^s = \sum_{n=0}^{\infty} \binom{s}{n} x^n, \qquad |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \qquad |x| < 1$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + r_n(x), \qquad \text{where}$$

$$r_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(t_x) x^{n+1}, \qquad 0 < t_x < x$$

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1. $\lim_{n\to\infty}\left(1+\frac{2}{3n}\right)^{4n} = \quad 1^{\infty}$

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{2}{3n}\right)^{4n}} = e^{\lim_{n\to\infty}\ln\left(1+\frac{2}{3n}\right)^{4n}} \overset{(*)}{=} e^{8/3}$$

A. e^4

B. $e^{8/3}$

C. e^8

D. $e^{2/3}$

E. $e^{1/6}$

$$(*) \lim_{n\to\infty} \ln\left(1+\frac{2}{3n}\right)^{4n} = \lim_{n\to\infty} (4n)\ln\left(1+\frac{2}{3n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{3n}\right)}{\frac{1}{4n}} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{2}{3n}}\right)\left(\frac{2}{3}\right)\left(\frac{-1}{n^2}\right)}{\frac{1}{4}\left(\frac{-1}{n^2}\right)}$$

$$= \frac{(1)(2/3)}{1/4} = \frac{8}{3}$$

2. Which of the following statements are true?

False I. If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

note: $\lim_{n\to\infty} b_n \neq 0$

False II. If the sequence $\{a_n\}_{n=m}^{\infty}$ is bounded, then it is convergent.

$\lim_{n\to\infty} \sin n$ does not exist, yet $\sin n$ is bounded

False III. $\lim_{n\to\infty} \sqrt[n]{en}\left(\frac{2n+1}{n-162}\right) = 2e.$

$= (1)(2) = 2.$

A. Only I and II

B. Only I

C. Only II

D. Only II and III

E. None

3. $\sum_{n=1}^{\infty}\frac{3^n-5^n}{15^n} = \sum_{n=1}^{\infty}\frac{3^n}{15^n} - \sum_{n=1}^{\infty}\frac{5^n}{15^n}$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n \quad \text{(both series converge)}$$

$$= \frac{\frac{1}{5}}{1-\frac{1}{5}} - \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{\frac{1}{5}}{\frac{4}{5}} - \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

A. $\frac{8}{7}$

B. $-\frac{8}{7}$

C. $-\frac{2}{13}$

D. $\frac{2}{13}$

E. $-\frac{1}{4}$

4. The series $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

A. converges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$

B. converges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$

C. diverges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$

D. diverges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$

E. diverges by the integral test

$\frac{1}{n3^n} < \frac{1}{3^n}$ for $n > 1$

and $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges.

5. Apply the ratio test to the series $\sum_{n=2}^{\infty} (-1)^n \frac{n+1}{n^3-1}$. The ratio test indicates

A. the series diverges

B. the series converges conditionally

C. the series converges absolutely

D. the series converges

E. no conclusion

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|a_{n+1} \cdot \frac{1}{a_n}\right|$$

$$= \left|\frac{n+2}{(n+1)^3-1} \cdot \frac{n^3-1}{n+1}\right|$$

$\longrightarrow \underline{1}$ as $n \to \infty$

Therefore there is no conclusion.

6. Consider these two alternating series:

(I) $\sum_{n=2}^{\infty} \left(-\frac{3}{4}\right)^n \left(\frac{n-1}{n+1}\right)$

(II) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

A. (I) diverges and (II) converges absolutely

B. (I) diverges and (II) converges conditionally

C. (I) converges conditionally and (II) diverges

D. (I) converges absolutely and (II) converges conditionally

E. (I) converges conditionally and (II) converges conditionally

(I) $\sum^{\infty} \left(\frac{3}{4}\right)^n$ converges and

$\left(\frac{3}{4}\right)^n \left(\frac{n-1}{n+1}\right) < \left(\frac{3}{4}\right)^3$. Therefore

$\sum^{\infty} \left(-\frac{3}{4}\right)^n \left(\frac{n-1}{n+1}\right)$ converges absolutely.

II $\sum^{\infty} \frac{(-1)^n}{\sqrt{n}}$ convergent alternating series. $\sum^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series.

Therefore $\sum^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

7. The interval of convergence of

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^n}{n^2} \text{ is}$$

A. $[-3, 3]$
B. $[-2, 2]$
C. $\left[-\frac{2}{3}, \frac{2}{3}\right)$
D. $\left[-\frac{3}{2}, \frac{3}{2}\right]$
E. $\left[-\frac{3}{2}, \frac{3}{2}\right)$

Root Test: $\lim_{n\to\infty} \left| \left(\frac{2}{3}\right)^n \frac{x^n}{n^2} \right|^{1/n}$

$= \lim_{n\to\infty} \left(\frac{2}{3}\right) |x| \frac{1}{(n^{1/n})^2} = \frac{2}{3} |x|.$

$\frac{2}{3}|x| < 1 \longrightarrow |x| < 3/2 \longrightarrow -3/2 < x < 3/2.$

endpts: $x = -\frac{3}{2} \rightarrow \sum^{\infty} \frac{(-1)^n}{n^2}$ converges.

$x = \frac{3}{2} \rightarrow \sum^{\infty} \frac{1}{n^2}$ converges.

8. Use the Taylor series of $\sin(x^2)$ to approximate $\int_0^1 \sin(x^2)dx$ with error less than 0.001. The smallest number of nonzero terms of the series that are needed for this accuracy is

A. 1
B. 2
C. 3
D. 4
E. 5

$\int_0^1 \sin(x^2)dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \right) dx$

$= \frac{1}{3}x^3 \Big|_0^1 - \frac{x^7}{7\cdot 3!}\Big|_0^1 + \frac{x^{11}}{11\cdot 5!}\Big|_0^1 - \cdots$

$= \underbrace{\frac{1}{3} - \frac{1}{42}} + \frac{1}{(11)(120)} - \cdots$

1^{st} term $< \frac{1}{1000}$

Therefore smallest number of terms is 2.

9. The radius of convergence of $\sum_{n=1}^{\infty} \frac{10^n}{n!} x^n$ is

A. 10
B. 0
C. ∞
D. 1
E. $\frac{1}{10}$

Ratio Test: $\left| a_{n+1} \cdot \frac{1}{a_n} \right| = \left| \frac{10^{n+1}}{(n+1)!} x^{n+1} \cdot \frac{n!}{10^n \cdot x^n} \right|$

$= 10 \cdot \frac{1}{n+1} \cdot |x| \longrightarrow 0$ as $n \to \infty$ for all x. Therefore series converges for all x.

Therefore radius of convergence is ∞

10. The Taylor series of

$$f(x) = \begin{cases} \dfrac{e^{x^2}-1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is given by

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots$$

$$e^{x^2}-1 = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots$$

$$\frac{e^{x^2}-1}{x} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!} = x + \frac{x^3}{2!} + \frac{x^5}{3!} + \cdots$$

A. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)!}$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)!}$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(n-1)!}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n-1}}{n!}$

E. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n!}$

11. $(1-x^2)^{\frac{3}{2}} =$ (use binomial series)

$$\left(1 + (-x^2)\right)^{3/2}$$

$$= 1 + \frac{3}{2}(-x^2) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x^2)^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(-x^2)^3 + \cdots$$

$$= 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6 + \cdots$$

A. $1 + \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{3}{16}x^6 \ldots$

B. $1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6 \ldots$

C. $1 - \frac{3}{2}x^2 + \frac{3}{4}x^4 + \frac{1}{8}x^6 \ldots$

D. $1 - \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{8}x^3 \ldots$

E. $1 - \frac{3}{2}x + \frac{3}{4}x^2 - \frac{1}{8}x^3 \ldots$

12. If $\begin{aligned} x &= -2 + \frac{1}{2}\cos(2t) \\ y &= 1 - \frac{1}{2}\sin(2t) \end{aligned}$ then the equation relating x and y is given by

$$(x+2)^2 + (y-1)^2$$

$$= \left(\frac{1}{2}\cos(2t)\right)^2 + \left(-\frac{1}{2}\sin(2t)\right)^2$$

$$= \frac{1}{4}$$

A. $x^2 + y^2 = \frac{1}{4}$

B. $(x-y-3)^2 = 1$

C. $(x+2)^2 + (y-1)^2 = \frac{1}{4}$

D. $(x+2)^2 + (y-1)^2 = 4$

E. $(x-2)^2 + (y+1)^2 = 1$