Math 265 Linear Algebra — Final Exam — Fall 2000

Student Name (print):	Student ID:

Circle the name of your instructor (with the time of your class):

Ban	Dugger	Gottlieb	Stefanov
Hulanicki (1:30)	Hulanicki (2:30)	Matsuki (12:30)	Matsuki (1:30)
Pascovici (8:30)	Pascovici (9:30)	Shipley (10:30)	Shipley (3:00)
Włodarczyk (9:00)	Włodarczyk (10:30)		

Do not write below this line.

Please be neat and show all work.
Write each answer in the provided box.
Use the back of the sheets and the last 3 pages for extra scratch space.
Return this entire booklet to your instructor.
No books. No notes. No calculators.

Problem #	Max pts.	Earned points
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
Section I	**100**	

12	10	
13	10	
14	10	
15	10	
Section II	**40**	
16	20	
17	20	
18	20	
Section III	**60**	
TOTAL	**200**	

Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that $A = \begin{bmatrix} 1 & 3 & 4 & 1 & 4 \\ 2 & 3 & 2 & 2 & 5 \\ 0 & 2 & 4 & 0 & 2 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}$ and $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find the nullity of A.

(c) Find a basis for the column space of A.

(d) Find a basis for the row space of A.

(e) Find a basis for the null space of A.

2. Let $A = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}$ be a 2×2 matrix and let $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.
Determine the value(s) of x so that $AB = BA$.

3. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

4. Determine the value(s) of a so that the line whose parametric equations are given by

$$\begin{cases} x = 2 + 3t \\ y = 3 - t \\ z = 1 + at \end{cases}$$

and the plane

$$2x - y + z + 3 = 0$$

do NOT intersect.

5. Find the value(s) of k for which the vector $v = \begin{bmatrix} 1 \\ 6 \\ k \end{bmatrix}$ is in the space spanned by

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix} \right\}.$$

6. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

7. E is a 3×3 matrix of the form

$$E = \begin{bmatrix} 0 & 1 & 1 \\ x & y & z \\ 3 & 5 & 7 \end{bmatrix}.$$

Given $\det(E) = 3$, compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 0 & 1 & 1 \\ 6+3x & 10+3y & 14+3z \end{bmatrix}$$

8. Find a diagonal matrix similar to $A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$.

9. Find the projection $\mathrm{Proj}_{\mathrm{W}}\mathrm{v}$ of the vector $v = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$ onto the subspace W spanned by

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

10. Let v and w be two vectors in $\mathbb{R}^5$ such that $v \cdot v = 4, w \cdot w = 9$ and $v \cdot w = 1$. Find the cosine of the angle between v and w.

11. We have a subspace W in $\mathbb{R}^4$ spanned by the following three linearly independent vectors

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Find an orthonormal basis of W.

Section II: Multiple choice problems

For Problems 12 through 15, circle only one (the correct) answer for each part. No partial credit.

12. Let A be a 3×5 matrix with $rank(A) = 2$. Determine if each of the following statements is true or false.

(a) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. True False

(b) If the rank of the augmented matrix $[A\ \mathbf{b}]$ is also 2, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. True False

(c) If the rank of the augmented matrix $[A\ \mathbf{b}]$ is 3, then the system $A\mathbf{x} = \mathbf{b}$ has no solution. True False

(d) The system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A .

True False

(e) The null space of A has dimension 2. True False

13. Determine if each of the following statements is true or false, where A is a 3×3 matrix.

(a) If 2 is an eigenvalue for A, then $2I_3 - A$ is singular. True False

(b) If $A^2 = A$, then $\det(A) = 0$ or 1. True False

(c) $\det(kA) = k\det(A)$ for any scalar k. True False

(d) If the characteristic polynomial of A has roots 1, 2 and 3, then A is diagonalizable.

True False

(e) If A is nonsingular, then none of the eigenvalues of A is 0. True False

14. For each of the following sets, determine if it is a vector (sub)space:

(a) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1 - x_2 + x_3 - x_4 = 0$;

Yes No

(b) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1 x_2 - x_3 x_4 = 0$;

Yes No

(c) The set of all vectors $v = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with length being equal to 1;

Yes No

(d) The set of all vectors of the form $v = (a+b, 2a+3c, b-c, a+b+c)$ in $\mathbb{R}^4$ where a, b and c are arbitrary real numbers;

Yes No

(e) The set of all solutions to the linear system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$;

Yes No

15. For the problems (a), (b) and (c), determine if the given set of vectors is linearly independent or linearly dependent:

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$; Independent Dependent

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$; Independent Dependent

(c) $\left\{ \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}$; Independent Dependent

For the problems (d) and (e), determine if the given set of vectors spans $\mathbb{R}^3$:

(d) $\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$; span not span

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$; span not span

Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

16. Find the least squares fit line for the points

$$(-1, 3), (0, 2), (1, 4).$$

17. Let

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for A.

Note: The eigenvalues are COMPLEX-valued.

(b) Find the general REAL solution to the linear system of differential equations.

18. Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the linear system of differential equations where

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for B.

(b) Find the general solution to the linear system of differential equations