

Student Name (print):

Student ID:

Circle the name of your instructor (with the time of your class):

- |                   |                    |                 |                |
|-------------------|--------------------|-----------------|----------------|
| Ban               | Dugger             | Gottlieb        | Stefanov       |
| Hulanicki (1:30)  | Hulanicki (2:30)   | Matsuki (12:30) | Matsuki (1:30) |
| Pascovici (8:30)  | Pascovici (9:30)   | Shiple (10:30)  | Shiple (3:00)  |
| Włodarczyk (9:00) | Włodarczyk (10:30) |                 |                |

Do not write below this line.

Please be neat and show all work.  
 Write each answer in the provided box.  
 Use the back of the sheets and the last 3 pages for extra scratch space.  
 Return this entire booklet to your instructor.  
**No books. No notes. No calculators.**

Problem #	Max pts.	Earned points
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
<b>Section I</b>	<b>100</b>	

12	10	
13	10	
14	10	
15	10	
<b>Section II</b>	<b>40</b>	
16	20	
17	20	
18	20	
<b>Section III</b>	<b>60</b>	
<b>TOTAL</b>	<b>200</b>	

## Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that  $A = \begin{bmatrix} 1 & 3 & 4 & 1 & 4 \\ 2 & 3 & 2 & 2 & 5 \\ 0 & 2 & 4 & 0 & 2 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}$  and  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

(a) Find the rank of  $A$ .

(b) Find the nullity of  $A$ .

(c) Find a basis for the column space of  $A$ .

(d) Find a basis for the row space of  $A$ .

(e) Find a basis for the null space of  $A$ .

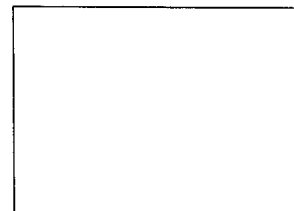
2. Let  $A = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}$  be a  $2 \times 2$  matrix and let  $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ .  
Determine the value(s) of  $x$  so that  $AB = BA$ .



3. Let  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Find  $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .



4. Determine the value(s) of  $a$  so that the line whose parametric equations are given by

$$\begin{cases} x = 2 + 3t \\ y = 3 - t \\ z = 1 + at \end{cases}$$

and the plane

$$2x - y + z + 3 = 0$$

do NOT intersect.

5. Find the value(s) of  $k$  for which the vector  $v = \begin{bmatrix} 1 \\ 6 \\ k \end{bmatrix}$  is in the space spanned by

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix} \right\}.$$

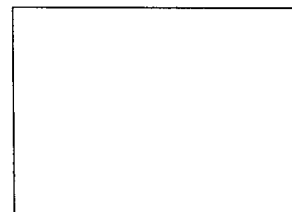
6. Compute the inverse of the matrix  $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ .

7.  $E$  is a  $3 \times 3$  matrix of the form

$$E = \begin{bmatrix} 0 & 1 & 1 \\ x & y & z \\ 3 & 5 & 7 \end{bmatrix}.$$

Given  $\det(E) = 3$ , compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 0 & 1 & 1 \\ 6 + 3x & 10 + 3y & 14 + 3z \end{bmatrix}$$

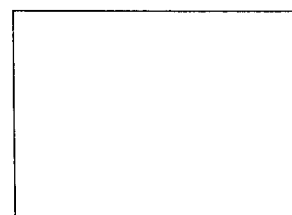


8. Find a diagonal matrix similar to  $A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$ .



9. Find the projection  $\text{Proj}_W v$  of the vector  $v = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$  onto the subspace  $W$  spanned by

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$



10. Let  $v$  and  $w$  be two vectors in  $\mathbb{R}^5$  such that  $v \cdot v = 4$ ,  $w \cdot w = 9$  and  $v \cdot w = 1$ . Find the cosine of the angle between  $v$  and  $w$ .

11. We have a subspace  $W$  in  $\mathbb{R}^4$  spanned by the following three linearly independent vectors

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Find an orthonormal basis of  $W$ .

## Section II: Multiple choice problems

For Problems 12 through 15, circle only one (the correct) answer for each part.  
No partial credit.

12. Let  $A$  be a  $3 \times 5$  matrix with  $\text{rank}(A) = 2$ . Determine if each of the following statements is true or false.
- (a) For some  $\mathbf{b}$ , the system  $A\mathbf{x} = \mathbf{b}$  has a unique solution. True   False
  - (b) If the rank of the augmented matrix  $[A \ \mathbf{b}]$  is also 2, then the system  $A\mathbf{x} = \mathbf{b}$  has infinitely many solutions. True   False
  - (c) If the rank of the augmented matrix  $[A \ \mathbf{b}]$  is 3, then the system  $A\mathbf{x} = \mathbf{b}$  has no solution. True   False
  - (d) The system  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if  $\mathbf{b}$  is in the column space of  $A$ . True   False
  - (e) The null space of  $A$  has dimension 2. True   False
13. Determine if each of the following statements is true or false, where  $A$  is a  $3 \times 3$  matrix.
- (a) If 2 is an eigenvalue for  $A$ , then  $2I_3 - A$  is singular. True   False
  - (b) If  $A^2 = A$ , then  $\det(A) = 0$  or 1. True   False
  - (c)  $\det(kA) = k \det(A)$  for any scalar  $k$ . True   False
  - (d) If the characteristic polynomial of  $A$  has roots 1, 2 and 3, then  $A$  is diagonalizable. True   False
  - (e) If  $A$  is nonsingular, then none of the eigenvalues of  $A$  is 0. True   False

14. For each of the following sets, determine if it is a vector (sub)space:

(a) The set of all vectors  $(x_1, x_2, x_3, x_4)$  in  $\mathbb{R}^4$  with the property  $x_1 - x_2 + x_3 - x_4 = 0$ ;  
Yes No

(b) The set of all vectors  $(x_1, x_2, x_3, x_4)$  in  $\mathbb{R}^4$  with the property  $x_1x_2 - x_3x_4 = 0$ ;  
Yes No

(c) The set of all vectors  $v = (x_1, x_2, x_3)$  in  $\mathbb{R}^3$  with length being equal to 1;  
Yes No

(d) The set of all vectors of the form  $v = (a + b, 2a + 3c, b - c, a + b + c)$  in  $\mathbb{R}^4$  where  $a, b$  and  $c$  are arbitrary real numbers;  
Yes No

(e) The set of all solutions to the linear system of differential equations  $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$   
where  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ;  
Yes No



15. For the problems (a), (b) and (c), determine if the given set of vectors is linearly independent or linearly dependent:

$$(a) \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

$$(b) \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

$$(c) \left\{ \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

For the problems (d) and (e), determine if the given set of vectors spans  $\mathbb{R}^3$ :

$$(d) \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}; \quad \text{span} \quad \text{not span}$$

$$(e) \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}; \quad \text{span} \quad \text{not span}$$

### Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

16. Find the least squares fit line for the points

$$(-1, 3), (0, 2), (1, 4).$$

17. Let

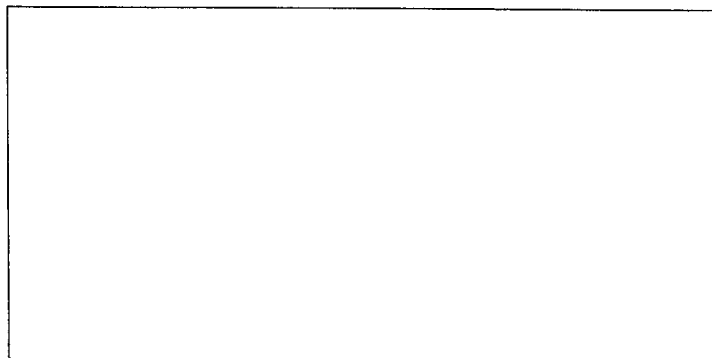
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

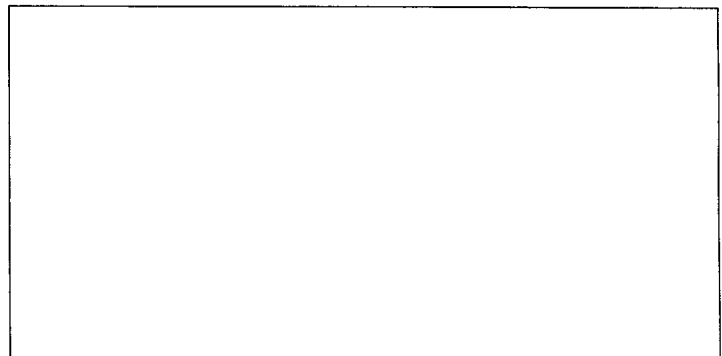
$$A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for  $A$ .

Note: The eigenvalues are COMPLEX-valued.



(b) Find the general REAL solution to the linear system of differential equations.



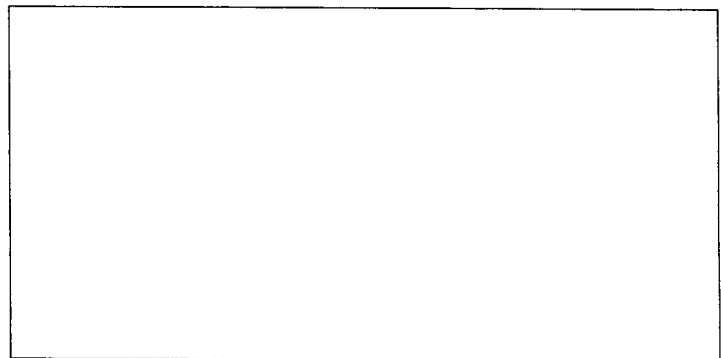
18. Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the linear system of differential equations where

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for  $B$ .



(b) Find the general solution to the linear system of differential equations

