MA 266 Summer 01 REVIEW 1

DEFINITION OF SOLUTION

A function $y = y(t)$ is a solution of the differential equation $\dfrac{dy}{dt} = f(t, y)$ on the interval $\alpha < t < \beta$ if $\dfrac{dy}{dt}(t) = f(t, y(t))$ for $\alpha < t < \beta$.

A function $y = y(t)$ is a solution of the initial value problem $\dfrac{dy}{dt} = f(t, y),\ y(t_0) = y_0$, if $y = y(t)$ is a solution of the differential equation $\dfrac{dy}{dt} = f(t, y)$ on some interval $\alpha < t < \beta$ that contains t_0 and $y(t_0) = y_0$.

DIRECTION FIELD

A solution of the differential equation $y' = f(t, y)$ has slope $f(t_0, y_0)$ at the point (t_0, y_0). The direction field of a differential equation $y' = f(t, y)$ indicates the slope of solutions at various points (t_0, y_0).

The direction field gives information about the behavior of solutions that correspond to different initial values.

The direction field may indicate a restriction on the domain of a solution if the graph of the solution appears to approach either a vertical asymptote $t = t_1$ or a point (t_1, y_1) where the graph becomes vertical.

Direction fields can be plotted using MATLAB and the program dfield.

We will be using the programs dfield, pplane, eul, rk2, and rk4 with MATLAB. These programs are not a part of MATLAB, but should be installed with MATLAB at all PUCC pc and Mac labs on campus. If you are using a student version of MATLAB on your own computer, you will need to download the files that are appropriate for your version of MATLAB. The files can be downloaded from http://math.rice.edu/~dfield/.

LINEAR EQUATIONS

$y' + p(t)y = g(t)$

Multiply by the integrating factor $\mu(t) = e^{\int p(t)\,dt}$, so $\mu' = \mu p$. Then

$\mu y' + \mu p y = \mu g$

$\mu y' + \mu' y = \mu g$

$(\mu y)' = \mu g$

$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C$

$$y = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)}$$

SEPARABLE EQUATIONS

$$N(y)\frac{dy}{dx} = M(x)$$

$$\int N(y(x))\frac{dy}{dx}\,dx = \int M(x)\,dx$$

$$\int N(y)\,dy = \int M(x)\,dx$$

Equations of the form $\dfrac{dy}{dx} = m(x)n(y)$ can be put into separable form by dividing by $n(y)$. In addition to the solution obtained by integration, the differential equation will have constant solutions $y = y_0$, where y_0 is a solution of $n(y) = 0$.

You should know how to solve first order differential equations that are either separable or linear.
You should be able to evaluate integrals of the following types:

$\int$(polymonial) dx,
$\int e^{rx}\,dx$,
$\int (ax+b)^r\,dx$, (including $r = -1$)
$\int \dfrac{ax+b}{(x-r_1)(x-r_2)}\,dx$, (partial fractions)

You should be able to use given values $y(x_0) = y_0$ to determine unknown constants in a solution.
You should know the relation of the graph of the solution of an initial value problem to the corresponding direction field.
MA 266 SPR 01 REVIEW 1 PRACTICE QUESTIONS

1. Determine the order of each of the differential equations; also state whether the equation is linear or nonlinear.

(a) $yy' + t = 1$
(b) $ty' + y = 1$
(c) $(y')^3 + ty = 1$
(d) $y''' + \sqrt{t}y = 1$

2. (a) Which of the functions $y_1(t) = t$ and $y_2(t) = -t$ are solutions of the initial value problem $yy' = t,\ y(0) = 0$?

(b) Which of the functions $y_1(t) = t$ and $y_2(t) = -t$ are solutions of the initial value problem $yy' = t,\ y(1) = 1$?

3. (a) Show that $y = t^3$ is a solution of the initial value problem
$y' = 3y^{2/3},\ y(0) = 0$.

(b) Find a different solution of the initial value problem.

4. For what value(s) of r is $y = e^{rt}$ a solution of the differential equation $y'' - 5y' + 6y = 0$?

5. Find the general solution of the differential equation $y' + \dfrac{1}{t+1}y = 1$. Assume $t + 1 > 0$.

6. Find the solution of the initial value problem $ty' = y + 1,\ y(1) = 2$.

7. For what value(s) of a is the solution of the initial value problem $y' - y + 2e^{-t} = 0,\ y(0) = a$ bounded on the interval $t \geq 0$?

8. Use the given direction field of $y' = (y-1)(y-3)$ to determine the behavior of y as t increases for each initial value $y(0) = a$.

9. Use the given direction fields to sketch the solution of the corresponding initial value problem for the indicated initial value (t_0, y_0). Extend your sketch in both directions as far as seems possible and explain why the domain of the solution may be restricted.

(a) $(t_0, y_0) = (1, 1)$ (b) $(t_0, y_0) = (0, -1)$

10. Sketch the direction field of (a) $y' = -\dfrac{y}{t}$ and (b) $y' = -\dfrac{t}{y}$.

11. Find an implicit solution of the initial value problem $y' = \dfrac{1-2t}{1+3y^2}, y(1) = 2$.

12. Find an explicit solution of the initial value problem $t^2 y' = y^2,\ y(1) = 1/2$. Indicate the interval in which the solution is valid.

13 Find the slope of the solution of the differential equation $y' = 2y^3 + 4t$ at the point $(2, -1)$.

MA 266 SPR 01 REVIEW 1 PRACTICE QUESTION ANSWERS

1. (a) first order, nonlinear, (b) first order, linear, (c) first order, nonlinear, (d) third order, linear

2. (a) y_1 and y_2, (b) y_1

3. (a) $y' = 3t^2 = 3\left(t^3\right)^{2/3} = 3y^{2/3}$, $(0)^3 = 0$, (b) $y = 0$

4. $r = 2, 3$

5. $y = \dfrac{t^2}{2(t+1)} + \dfrac{t}{t+1} + \dfrac{C}{t+1}$ or $y = \dfrac{t+1}{2} + \dfrac{C}{t+1}$

6. $y = -1 + 3t$

7. $a = 1$

8. $y \to \infty$ as t increases if $a > 3$; $y \to 3$ as t increases if $a = 3$; $y \to 1$ as t increases if $a < 3$.

9. (a) The graph becomes vertical near $(t, y) = (\pm 2, 0)$.

(b) The graph approaches a vertical asymptote near $t = 2$.

10. (a)

(b)

11. $y + y^3 = t - t^2 + 10$
12. $y = \dfrac{t}{t+1},\ t > -1$
13. slope$= 6$

MA 266 SPR 00 REVIEW 2

EXISTENCE AND UNIQUENESS

THEOREM: If p and g are continuous in an interval $\alpha < t < \beta$ and t_0 is in the interval, then the initial value problem $y' + p(t)y = g(t),\ y(t_0) = y_0$ has a unique solution on the interval $\alpha < t < \beta$ for each real number y_0.

THEOREM: If f and f_y are continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$, and (t_0, y_0) is in the rectangle, then the initial value problem
$y' = f(t, y),\ y(t_0) = y_0$ has a unique solution in some interval $t_0 - h < t < t_0 + h$.
The Theorem does not guarantee a solution for $\alpha < t < \beta$.

EQUATIONS OF THE FORM $\dfrac{dy}{dt} = F(y)$

You should be able to solve such equations, determine equibrium solutions, determine whether equilibrium solutions are stable, semistable, or unstable, interpret the graph of $(y, F(y))$ to obtain signs of $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2} = \dfrac{dF}{dy} \cdot F(y)$, and sketch solutions of corresponding initial value problems. Solutions $y = y(t)$ are increasing functions of t in intervals where $F(y) > 0$ and decreasing functions of t where $F(y) < 0$. The graphs of solutions are concave upward in intervals where $\dfrac{dF}{dy} \cdot F(y) > 0$ and concave downward in intervals where $\dfrac{dF}{dy} \cdot F(y) < 0$.

SPECIAL SECOND ORDER DIFFERENTIAL EQUATIONS

You should be able to use the change of variables $v(t) = y'(t)$ to solve equations of the form $y'' = f(t, y')$. $(y'' = \dfrac{dv}{dt})$
You should be able to use the change of variables $v(y) = y'(t)$ to solve equations of the form $y'' = f(y, y')$. $(y'' = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy})$