FIRST ORDER DIFFERENTIAL EQUATIONS

You should be able to recognize and know how to solve first order differential equations that are either separable, linear, exact, or homogeneous.

You should be able to evaluate integrals of the following types:

$\int (\text{polymonial}) \, dx$,

$\int e^u \, du$,

$\int u^r \, du$, (including $r = -1$)

$\displaystyle\int \frac{ax + b}{(x - r_1)(x - r_2)} \, dx$, (partial fractions)

You should be able to use given values $y(x_0) = y_0$ to determine unknown constants in a solution.

You should know the relation of the graph of the solution of an initial value problem to the corresponding direction field.

HOMOGENEOUS EQUATIONS

$\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$

Let $y = xv$, so $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$ and $\dfrac{y}{x} = v$.

Substitute to obtain $x\dfrac{dv}{dx} + v = F(v)$.

Solve the above separable equation for v in terms of x.

Substitute $v = \dfrac{y}{x}$ to obtain a formula for the solution y of the original homogeneous equation.

EXACT EQUATIONS

$M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$ is exact if $M_y(x, y) = N_x(x, y)$.

Find a function $\psi(x, y)$ such that $\psi_x(x, y) = M(x, y)$ and $\psi_y(x, y) = N(x, y)$.

$\left(\psi(x, y) = \int M(x, y) \, dx + h(y); \text{ solve } \dfrac{\partial}{\partial y}\left(\int M(x, y) \, dx\right) + h'(y) = N(x, y)\right.$

for $h(y)$.)

A solution $y = y(x)$ of the exact equation then satisfies

$\dfrac{d}{dx}(\psi(x, y)) = \psi_x(x, y) + \psi_y(x, y)\dfrac{dy}{dx} = M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$,

so the general solution is of the form $\psi(x, y) = c$.

Numerical Methods For Solving $y' = f(t,y), \ y(t_0) = y_0$**:**

Create an M–file to define the function $f(t,y)$. The function name and the file name should be the same. Note that M–files are not entered in the matlab *command* window, but are external text files that are created with a text editor.

EXAMPLE   If $y' = \sqrt{t+y}$, create an M–file named f11.m:

  function z=f11(t,y)

  z=sqrt(t+y);

The general syntax for the Euler tangent line method is

$\gg$ [t,y]=eul('dfile',t0,tfinal,y0,stepsize);

Note that stepsize $=$ (tfinal $-$ t0)$/n$, where n is the number of steps.

EXAMPLE   To find the Euler tangent line approximation of the solution of the initial value problem $y' = \sqrt{t+y}, \ y(1) = 3$, where $t = 2$ using stepsize $h = 0.5$:

$\gg$ [t,y]=eul('f11',1,2,3,0.5);

$\gg$ [t,y]

  ans=

1.0000	3.0000
1.5000	4.0000
2.0000	5.1726

The syntax is the same for the improved Euler method (use rk2 in place of eul) and the runge–kutta method (use rk4 in place of eul.)

To obtain the graph of an approximate solution on a direction field, enter

$\gg$plot(t,y,C)

where C='o','x','+'. Omit C for a connected graph.

APPROXIMATE SOLUTIONS

The matlab commands eul, rk2, and rk4 can be used to obtain approximate solutions of the initial value problem $y' = f(x, y), \ y(x_0) = y_0$.

You should be able to use the formula $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h$ to evaluate Euler tangent line values by hand.

Approximation methods may not give good approximations of the solution of the initial value problem $y' = f(x, y), \ y(x_0) = y_0$, if:

• The initial value problem does not have a unique solution, because either f or f_y is not continuous at the initial point.

• The approximation extends beyond the interval where the solution is valid, because either $y'(t)$ or $y(x)$ becomes unbounded.

• The solution is unstable, because solutions that have slightly different initial values diverge from the desired solution.

PROPERTIES OF SOLUTIONS

If f has continuous first partial derivatives, then solutions of the differential equation $y' = f(x, y)$ satisfy

$$y'' = f_x(x, y) + f_y(x, y)\frac{dy}{dx}(x) = f_x(x, y) + f_y(x, y)f(x, y).$$

If $y(x)$ is a solution of the differentiable equation $y' = f(x, y)$:

If $y' > 0$ at a point, then y is increasing near the point.

If $y' < 0$ at a point, then y is decreasion near the point.

If $y'' > 0$ at a point, then y is concave upward and the Euler tangent line approximations are less than or equal to the solution near the point.

If $y'' < 0$ at a point, then y is concave downward and the Euler tangent line approximations are greater than or equal to the solution near the point.

The Taylor expansion of f about $x = c$ is

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \frac{f''''(c)}{4!}(x - c)^4 + \cdots$$

MA 266   SPR 00   REVIEW 3 PRACTICE QUESTIONS

1. Determine whether each of the following differential equations is separable, homogeneous, linear, or exact. Briefly justify your answers.

(a) $2x + y + (x + 3y)\dfrac{dy}{dx} = 0$

(b) $x + 3y + (2x + y)\dfrac{dy}{dx} = 0$

(c) $x + 3y + 1 + (2x + y + 1)\dfrac{dy}{dx} = 0$

(d) $2xy + 1 + (x^2 + 1)\dfrac{dy}{dx} = 0$

(e) $x^2 + 1 + (y^2 + 1)\dfrac{dy}{dx} = 0$

2. Find the explicit solution of the initial value problem $y' = y^2 - 1$, $y(0) = 0$.

3. Find the general solution of the differential equation $xy' + 2y = x^2$.

4. Use the formula $y = xv$ to express the differential equation $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$ in terms of x, v, and $\dfrac{dv}{dx}$.

5. Find an implicit form of the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$.

6. Find an implicit solution of the initial value problem

$$2xy + 1 + (x^2 + 2y)\frac{dy}{dx} = 0, \quad y(1) = -1.$$

7. Determine approximate values at $x = 0.5$ of the solution of the initial value problem $y' = 3x + y$, $y(0) = 1$ by using the Euler tangent line method with $h = 0.25$.