MA 266 Summer 2001 REVIEW 5

## VARIATION OF PARAMETERS

If $y_1$ and $y_2$ are solutions of $y''+py'+qy=0$ with $W(y_1,y_2)\neq 0$, we can find a particular solution of $y''+py'+qy=g$ of the form $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$. The simplifying assumption (1) $u_1'y_1+u_2'y_2=0$ gives $y'=u_1y_1'+u_2y_2'$. Then substitution of $y, y'$, and $y''$ into $y''+py'+qy=g$ and simplifying gives the differential equation (2) $u_1'y_1'+u_2'y_2'=g$. Equations (1) and (2) can be solved for $u_1'$ and $u_2'$. This gives the following:

THEOREM: If $p, q$, and $g$ are continuous on an open interval $I$, $y_1$ and $y_2$ are solutions of the homogeneous differential equation $y''+p(t)y'+q(t)y=0$, and $W(y_1,y_2)\neq 0$, then the nonhomogeneous differential equation
$y''+p(t)y'+q(t)y=g(t)$ has particular solution
$$Y(t)=-y_1(t)\int\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt+y_2(t)\int\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt$$
and general solution $y=c_1y_1(t)+c_2y_2(t)+Y(t)$. ■

## HIGHER ORDER LINEAR EQUATIONS

The theory is similar to that for second order linear equations. This includes the interval in which solutions exist, the form of general solutions of homogeneous and nonhomogeneous equations with constant coefficients, and the method of undetermined coefficients.

## APPLICATIONS

Spring–mass system, $mu''+\gamma y'+ky=F_0\cos(\omega t)$:

The mass of an unforced system $mu''+\gamma y'+ky=0$ does not oscillate if the system is either critically damped, $\gamma=2\sqrt{km}$, or overdamped, $\gamma>2\sqrt{km}$. Otherwise, the system oscillates.

A forced, undamped system becomes unbounded as $t\to\infty$ if and only if $\omega=\sqrt{k/m}$.
A damped system is always bounded.

Know how to find steady–state solutions.

Know how to interpret initial conditions and graphs of solutions.

Know how to use the formulas $R\cos\delta=A,\ R\sin\delta=B,\ R=\sqrt{A^2+B^2}$, $A\cos(\omega_0t)+B\sin(\omega_0t)=R\cos(\omega_0t-\delta)$.

**SPRING–MASS SYSTEMS**

mass, $m = \dfrac{\text{Weight}}{g}$, $g = 32 \text{ ft/sec}^2 = 9.8 \text{ m/sec}^2 = 980 \text{ cm/sec}^2$,
gravitational force = weight = $mg$,
spring constant, $k = \dfrac{\text{Force}}{\text{Displacement}}$,
$u(t)$ = displacement from equilibrium position, where $mg = kL$,
spring force = $-k(L + u(t))$,
damping constant, $\gamma = \dfrac{\text{Force}}{\text{Speed}}$,
damping force = $-\gamma u'(t)$,
applied external force = $F(t)$,

$$\text{(mass)(acceleration)} = \text{(sum of all external forces)},$$

$$mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t),$$

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = F(t),\ u(0) = u_0,\ u'(0) = u_0',}$$

| System | Length | Mass | Time | Force |
|---|---|---|---|---|
| English | feet | slugs | seconds | pounds |
| mks | meters | kilograms | seconds | newtons |
| cgs | centimeters | grams | seconds | dynes |

$$\boxed{A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta),}$$

where

$$R\cos(\omega_0 t - \delta) = R(\cos(\omega_0 t)\cos\delta + \sin(\omega_0 t)\sin\delta),$$

so $R\cos\delta = A,\ R\sin\delta = B,\ R = \sqrt{A^2 + B^2},\ \tan\delta = B/A.$

MA 266 Spring 2001 REVIEW 5 PRACTICE QUESTIONS

In Problems 1–3 find the general solution of the homogeneous differential equations in (a) and use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equation in (b).

**1.** (a) $y''' - y' = 0$

(b) $y''' - y' = t + e^t$

**2.** (a) $y''' - y'' - y' + y = 0$

(b) $y''' - y'' - y' + y = e^t + \cos t$

**3.** (a) $y''' - y = 0$

(b) $y''' - y = te^{-t/2}\cos(\sqrt{3}t/2)$

**4.** Find the general solution of the differential equation $y''' + y' = t^2$.

**5.** Find the solution of the initial value problem
$y''' - 2y'' + y' = 0,\ y(0) = 2,\ y'(0) = 0,\ y''(0) = 1.$

**6.** (a) Find the general solution of the differential equation
$y'' + 5y' + 6y = 26\cos(2t)$.

(b) Find the steady–state solution of the differential equation
$y'' + 5y' + 6y = 26\cos(2t)$.

**7.** Use the formulas $R\cos\delta = A,\ R\sin\delta = B,\ R = \sqrt{A^2 + B^2}$,
$A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$ to find $R$ and $\delta$ such that
$-3\cos(2t) + 4\sin(2t) = R\cos(2t - \delta)$.

**8.** For what nonnegative values of $m$ will the the solution of the initial value problem $mu'' + 4u = 8\cos(4t),\ u(0) = 4,\ u'(0) = 0,$ become unbounded as $t \to \infty$?

**9.** For what nonnegative values of $\gamma$ will the the solution of the initial value problem $u'' + \gamma u' + 4u = 0,\ u(0) = 4,\ u'(0) = 0,$ oscillate?

**10.** A mass that weighs 4 pounds stretches a spring 0.25 feet. The mass is acted upon by an external force of $2\cos t$ pounds and moves in a medium that imparts a viscous force of 6 pounds when the speed of the mass is 3 feet/sec. At time $t = 0$ the mass is 0.5 feet below the equilibrium position of the system and the mass is moving upward at 5 feet/sec. **Set up** an initial value problem that describes the motion of the mass. You do not need to solve the initial value problem.

**11.** The differential equation $t^2y'' - ty' = 0$ has solutions $y_1 = 1$ and $y_2 = t^2$. Use the method of variation of parameters to find a solution of $t^2y'' - ty' = 4t^4$.

**12.** The differential equation $t^2y'' - 2ty' + 2y = 0$ has solution $y_1 = t$. Find the general solution of $t^2y'' - 2ty' + 2y = 2t^2$.

MA 266 Spring 2001 REVIEW 5 PRACTICE QUESTION ANSWERS

**1.** (a) $y = C_1 + C_2 e^{-t} + C_3 e^t$

(b) $y = t(At + B) + Cte^t$

**2.** (a) $y = C_1 e^t + C_2 te^t + C_3 e^{-t}$

(b) $y = At^2 e^t + B\cos t + C\sin t$

**3.** (a) $y = C_1 e^t + C_2 e^{-t/2}\cos(\sqrt{3}t/2) + C_3 e^{-t/2}\sin(\sqrt{3}t/2)$

(b) $y = te^{-t/2}((At + B)\cos(\sqrt{3}t/2) + (Ct + D)\sin(\sqrt{3}t/2))$

**4.** $y = C_1 + C_2\cos t + C_3\sin t + \frac{1}{3}t^3 - 2t$

**5.** $y = 3 - e^t + te^t$

**6.** (a) $y = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{2}\cos(2t) + \frac{5}{2}\sin(2t)$

(b) $y(\text{steady–state}) = \frac{1}{2}\cos(2t) + \frac{5}{2}\sin(2t)$

**7.** $R = 5,\ \delta = \tan^{-1}(-4/3) + \pi \approx 2.214.$

**8.** $m = 1/4$

**9.** $0 \le \gamma < 4$

**10.** $\frac{1}{8}u'' + 2u' + 16u = 2\cos t,\ u(0) = \frac{1}{2},\ u'(0) = -5$

**11.** $y = u_1 + t^2 u_2 = -\frac{1}{2}t^4 + t^2 \cdot t^2 = \frac{1}{2}t^4$

**12.** $y = C_1 t + C_2 t^2 + 2t^2 \ln t$