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THEOREM 7.1.1 If $F_1, \ldots, F_n$ and all of the first–order partial derivatives, $\dfrac{\partial F_1}{\partial x_1}, \ldots, \dfrac{\partial F_1}{\partial x_n}, \ldots, \dfrac{\partial F_n}{\partial x_1}, \ldots, \dfrac{\partial F_n}{\partial x_n}$, are continuous in a region $R : \alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_1 < \beta_n$, and $(t, x_1^0, \ldots, x_n^0)$ is in that region, then there is an interval $|t - t_0| < h$ in which there is a unique solution, $x_1(t) = \phi_1(t), \ldots, x_n(t) = \phi_n(t)$, of the system of first–order differential equations,

$$\begin{cases} x_1' &= F_1(t, x_1 \ldots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1 \ldots, x_n), \end{cases}$$

that satisfies $x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0$.

THEOREM 7.1.2 If $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$, are continuous in an open interval $I : \alpha < t < \beta$, and t_0 is in that interval, then there is a unique solution, $x_1(t) = \phi_1(t), \ldots, x_n(t) = \phi_n(t)$, of the system of first–order linear differential equations,

$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t), \end{cases}$$

that satisfies $x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0$. The solution is valid for all of I.

UNDETERMINED COEFFICIENTS

The method can be used to solve systems of the form $\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}(t)$, where $\mathbf{P}$ is a constant matrix and the components of $\mathbf{g}$ are polynomial, exponential, or sinusoidal functions, or sums of products of these. The procedure is similar to that for linear second order differential equations. The coefficients are vectors in the case of systems. Also, if $\mathbf{g}(\mathrm{t}) = \mathbf{c}e^{\lambda t}$, where $\xi e^{\lambda t}$ is a solution of the corresponding homogeneous equation, try $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, instead of $\mathbf{a}te^{\lambda t}$.

MIXING PROBLEMS

Flow through several connected tanks leads to a system of linear differential equations. You should be able to set up and solve these problems.

To solve the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

first, solve the equation $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$ for eigenvalues λ.

For each eigenvalue λ, solution of the system of linear equations

$$\begin{Bmatrix} (a-\lambda)\xi_1 + b\xi_2 = 0 \\ c\xi_1 + (d-\lambda)\xi_2 = 0 \end{Bmatrix}$$

gives a corresponding eigenvector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$. The system of differential equations then has solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}$.

If there are two distinct real eigenvalues, the corresponding two solutions form a fundamental solution set.

If there is a complex eigenvalue, the real and imaginary parts of the corresponding complex solution form a fundamental solution set. Note that

$$e^{(\alpha+\beta i)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) \quad \text{and} \quad e^{(\alpha-\beta i)t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right).$$

You need to use only one of the two eigenvalues $\alpha + \beta i$ and $\alpha - \beta i$ to find a fundamental solution set.

If there is only one real eigenvalue, then a fundamental solution set is formed by the corresponding solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}$ and the solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} te^{\lambda t} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{\lambda t}, \quad \text{where } \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \text{ is a solution of the system}$$

$$\begin{Bmatrix} (a-\lambda)\eta_1 + b\eta_2 = \xi_1 \\ c\eta_1 + (d-\lambda)\eta_2 = \xi_2 \end{Bmatrix}.$$

This system has a solution in the case of a double root.

A solution of the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is of the form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. As t varies, the solution traces a *parametric curve* in the x_1x_2-plane, called a **trajectory** of the system. The slope of the trajectory in the x_1x_2-plane at a point on the trajectory is given by the formula for the slope of a parametric curve,

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{x_2'(t)}{x_1'(t)}.$$

Note that the formula $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$

can be used to find x_1' and x_2' at any point in the x_1x_2-plane. Also, the signs of x_1' and x_2' give the direction of the trajectory at the point. That is, x_1 is increasing as t increasing if $x_1' > 0$ and x_1 is decreasing as t increases if $x_1' < 0$.

EXAMPLE The trajectory of the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ at $(1, -2)$ satisfies

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

The slope is then $\dfrac{x_2'}{x_1'} = \dfrac{-3}{1} = -3$. $x_1' = 1 > 0$ implies x_1 is increasing and $x_2 = -3 < 0$ implies x_2 is decreasing. See graph.

The MATLAB command pplane can be used to find trajectories of a system of two first–order linear differential equations. The **Graph** menu on the **PPLANE Display** window can be used to plot the solution variables against the independent variable t.

MA 266 Spring 2001 REVIEW 7 PRACTICE PROBLEMS

1. Find the solution of the initial value problem

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

2. Find the general solution of the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

3. Find the general solution of the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

4. Find the general solution of $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t.$

5. Find the general solution of $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

6. Find the general solution of $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 6e^{-t} \\ 1 \end{pmatrix}.$

7. Express the differential equation $y''' - y'' + y' - y = t$ as a system of first–order differential equations.

8. The equations $x_1 = y,\ x_2 = y'$ transform the second–order equation $t^2y'' - 2ty' + 2y = 0$ into the first–order system

$$\begin{cases} x_1' = x_2, \\ t^2 x_2' - 2tx_2 + 2x_1 = 0, \end{cases}$$

which has solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} t^2 \\ 2t \end{pmatrix}.$$

Use the above information to find the solution of the initial value problem

$$t^2y'' - 2ty' + 2y = 0,\ y(1) = 1,\ y'(1) = 3.$$

9. Find the trajectories that corresponds to each of the given systems of differential equations:

(a) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\text{solution} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

(b) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\text{solution} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

(c) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\text{solution} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^t + C_2 \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t \right)$$

(d) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\text{solution} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}$$

10. Pure water flows into Tank 1 at a rate of 5 gal/min. The well–mixed solution from Tank 1 then flows at a rate of 5 gal/min into Tank 2. The solution in Tank 2 flows out at a rate of 5 gal/min. Set up and solve an initial value problem that gives the amount of salt in Tank 1, $x_1(t)$, and the amount of salt in Tank 2, $x_2(t)$, if Tank 1 initially holds 50 gallons of brine with concentration 1 lb/gal and Tank 2 initially holds 25 gallons of brine with concentration 3 lb/gal.

11. Tank 1 initially holds 50 gallons of brine with concentration 1 lb/gal and Tank 2 initially holds 25 gallons of brine with concentration 3 lb/gal. The solution in Tank 1 flows at a rate of 5 gal/min into Tank 2, while the solution in Tank 2 flows back to Tank 1 at a rate of 5 gal/min. Set up and solve an initial value problem that gives the amount of salt in Tank 1, $x_1(t)$, and the amount of salt in Tank 2, $x_2(t)$.

12. Find the slope of the trajectory in the x_1x_2-plane of the solution of the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ at the point $(1, -1)$ and sketch a short segment of the trajectory as it passes through the point $(1, -1)$. Your sketch should indicate the slope and direction of the trajectory at that point.

MA 266 Spring 2001 REVIEW 7 PRACTICE QUESTION ANSWERS

1. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$

2. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t + C_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} e^t$

3. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + C_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right)$

4. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

5. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

6. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

7. $\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_3 - x_2 + x_1 + t \end{cases}$

8. $y = -t + 2t^2$

9.(a) G, (b) B, (c) A, (d) C

10. $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1/10 & 0 \\ 1/10 & -1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix},$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 50 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/10} + 25 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/5}$

11. $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1/10 & 1/5 \\ 1/10 & -1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix},$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{125}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{100}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t/10}$

12. Slope $= -2$