CHAPTER 3

SECTION 1

3. To show that $C$ is a vector space we must show that all eight axioms are satisfied.

A1. $(a + bi) + (c + di) = (a + c) + (b + d)i$
   \[= (c + a) + (d + b)i\]
   \[= (c + di) + (a + bi)\]

A2. $(x + y) + z = [(x_1 + x_2 i) + (y_1 + y_2 i)] + (z_1 + z_2 i)$
   \[= (x_1 + y_1 + z_1) - (x_2 + y_2 + z_2)i\]
   \[= (x_1 + x_2 i) + [(y_1 + y_2 i) + (z_1 + z_2 i)]\]
   \[= x + (y + z)\]

A3. $(a + bi) + (0 + 0i) = (a + bi)$

A4. If $z = a + bi$ then define $-z = -a - bi$. It follows that
   \[z + (-z) = (a + bi) + (-a - bi) = 0 + 0i = 0\]

A5. $\alpha[(a + bi) + (c + di)] = (\alpha a + \alpha c) + (\alpha b + \alpha d)i$
   \[= \alpha(a + bi) + \alpha(c + di)\]

A6. $(\alpha + \beta)(a + bi) = (\alpha + \beta)a + (\alpha + \beta)bi$
   \[= \alpha(a + bi) + \beta(a + bi)\]

A7. $(\alpha \beta)(a + bi) = (\alpha \beta)a + (\alpha \beta)bi$
   \[= \alpha(\beta a + \beta bi)\]
4. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be arbitrary elements of $\mathbb{R}^{m \times n}$.

A1. Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for each $i$ and $j$ it follows that $A + B = B + A$.

A2. Since $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$

for each $i$ and $j$ it follows that $(A + B) + C = A + (B + C)$

A3. Let $O$ be the $m \times n$ matrix whose entries are all 0. If $M = A + O$ then

$m_{ij} = a_{ij} + 0 = a_{ij}$

Therefore $A + O = A$.

A4. Define $-A$ to be the matrix whose $ij$th entry is $-a_{ij}$. Since $a_{ij} + (-a_{ij}) = 0$

for each $i$ and $j$ it follows that $A + (-A) = O$.

A5. Since $\alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$

for each $i$ and $j$ it follows that $\alpha(A + B) = \alpha A + \alpha B$.

A6. Since $(\alpha + \beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}$

for each $i$ and $j$ it follows that $(\alpha + \beta)A = \alpha A + \beta A$.

A7. Since $(\alpha \beta)a_{ij} = \alpha (\beta a_{ij})$

for each $i$ and $j$ it follows that $(\alpha \beta)A = \alpha (\beta A)$.

A8. Since $1 \cdot a_{ij} = a_{ij}$

for each $i$ and $j$ it follows that $1A = A$.

5. Let $f$, $g$ and $h$ be arbitrary elements of $C[a, b]$.

A1. For all $x$ in $[a, b]$ \[(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x). \]

Therefore \[f + g = g + f\]
A2. For all \( x \) in \([a, b]\),

\[
[(f + g) - h](x) = (f + g)(x) + h(x) \\
= f(x) + g(x) + h(x) \\
= f(x) + (g + h)(x) \\
= [f + (g + h)](x)
\]

Therefore

\[
[(f + g) + h] = [f + (g + h)]
\]

A3. If \( z(x) \) is identically 0 on \([a, b]\), then for all \( x \) in \([a, b]\)

\[
(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x)
\]

Thus

\[
f + z = f
\]

A4. Define \(-f\) by

\[
(-f)(x) = -f(x) \quad \text{for all } x \text{ in } [a, b]
\]

Since

\[
(f + (-f))(x) = f(x) - f(x) = 0
\]

for all \( x \) in \([a, b]\) it follows that

\[
f + (-f) = z
\]

A5. For each \( x \) in \([a, b]\)

\[
[\alpha(f + g)](x) = \alpha f(x) + \alpha g(x) \\
= (\alpha f)(x) + (\alpha g)(x)
\]

Thus

\[
\alpha(f + g) = \alpha f + \alpha g
\]

A6. For each \( x \) in \([a, b]\)

\[
[(\alpha + \beta)f](x) = (\alpha + \beta)f(x) \\
= \alpha f(x) + \beta f(x) \\
= (\alpha f)(x) + (\beta f)(x)
\]

Therefore

\[
(\alpha + \beta)f = \alpha f + \beta f
\]

A7. For each \( x \) in \([a, b]\),

\[
[(\alpha\beta)f](x) = \alpha\beta f(x) = \alpha[\beta f(x)] = [(\alpha \beta)f](x)
\]

Therefore

\[
(\alpha \beta)f = \alpha(\beta f)
\]

A8. For each \( x \) in \([a, b]\)

\[
1f(x) = f(x)
\]

Therefore

\[
1f = f
\]
6. The proof is exactly the same as in Exercise 5.

9. (a) If \( y = \beta 0 \) then

\[
y + y = \beta 0 + \beta 0 = \beta (0 + 0) = \beta 0 = y
\]

and it follows that

\[
(y + y) + (-y) = y + (-y) \\
y + [y + (-y)] = 0 \\
y + 0 = 0 \\
y = 0
\]

(b) If \( \alpha x = 0 \) and \( \alpha \neq 0 \) then it follows from part (a), A7 and A8 that

\[
0 = \frac{1}{\alpha} \cdot 0 = \frac{1}{\alpha} (\alpha x) = \left( \frac{1}{\alpha} \right) x = 1x = x
\]

10. Axiom 6 fails to hold.

\[
(\alpha + \beta)x = ((\alpha + \beta)x_1, (\alpha + \beta)x_2) \\
\alpha x + \beta x = ((\alpha + \beta)x_1, 0)
\]

12. A1. \( x \oplus y = x \cdot y = y \cdot x = y \oplus x \)

A2. \( (x \oplus y) \oplus z = x \cdot y \cdot z = x \oplus (y \oplus z) \)

A3. \( x \oplus 1 = x \cdot 1 = x \)

\[
1 \oplus x = 1 \cdot x = x \\
\text{Therefore 1 is the zero vector.}
\]

A4. Let \(-x = -1 \circ x = x^{-1} = \frac{1}{x}\)

It follows that

\[
x \oplus (-x) = x \cdot \frac{1}{x} = 1 \quad \text{(the zero vector)}.
\]

Therefore \( \frac{1}{x} \) is the additive inverse of \( x \) for the operation \( \oplus \).

A5. \( \alpha \circ (x \oplus y) = (x \cdot y)^\alpha = x^\alpha \cdot y^\alpha \)

\( \alpha \circ x \oplus \alpha \circ y = x^\alpha \oplus y^\alpha = x^\alpha \cdot y^\alpha \)

A6. \( (\alpha + \beta) \circ x = x^{(\alpha + \beta)} = x^\alpha \cdot x^\beta \)

\( \alpha \circ x \oplus \beta \circ x = x^\alpha \oplus x^\beta = x^\alpha \cdot x^\beta \)

A7. \( (\alpha \beta) \circ x = x^{\alpha \beta} \)

\( \alpha \circ (\beta \circ x) = \alpha \circ x^\beta = (x^\beta)^\alpha = x^{\alpha \beta} \)

A8. \( 1 \circ x = x^1 = x \)

Since all eight axioms hold, \( R^+ \) is a vector space under the operations of \( \circ \) and \( \oplus \).

13. The system is not a vector space. Axioms A3, A4, A5, A6 all fail to hold.

14. Axioms 6 and 7 fail to hold. To see this consider the following example. If \( \alpha = 1.5, \beta = 1.8 \) and \( x = 1 \), then

\[
(\alpha + \beta) \circ x = [3.3] \cdot 1 = 3
\]
and
\[ \alpha \circ x + \beta \circ x = [1.5] \cdot 1 + [1.8] \cdot 1 = 1 \cdot 1 + 1 \cdot 1 = 2 \]

So Axiom 6 fails. Furthermore,
\[ (\alpha \beta) \circ x = [2.7] \cdot 1 = 2 \]

and
\[ \alpha \circ (\beta \circ x) = [1.5]([1.8] \cdot 1) = 1 \cdot (1 \cdot 1) = 1 \]

so Axiom 7 also fails to hold.

15. If \( \{a_n\}, \{b_n\}, \{c_n\} \) are arbitrary elements of \( S \), then for each \( n \)
\[ a_n + b_n = b_n + a_n \]

and
\[ a_n + (b_n + c_n) = (a_n + b_n) + c_n \]

Hence
\[ \{a_n\} + \{b_n\} = \{b_n\} + \{a_n\} \]
\[ \{a_n\} + ((\{b_n\} + \{c_n\}) = (\{a_n\} + \{b_n\}) + \{c_n\} \]

so Axioms 1 and 2 hold.

The zero vector is just the sequence \( \{0, 0, \ldots\} \) and the additive inverse
of \( \{a_n\} \) is the sequence \( \{-a_n\} \). The last four axioms all hold since
\[ \alpha(a_n + b_n) = \alpha a_n + \alpha b_n \]
\[ (\alpha + \beta)a_n = \alpha a_n + \beta a_n \]
\[ \alpha b_n = \alpha(\beta a_n) \]
\[ 1a_n = a_n \]

for each \( n \). Thus all eight axioms hold and hence \( S \) is a vector space.

16. If
\[ p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} \leftrightarrow a = (a_1, a_2, \ldots, a_n)^T \]
\[ q(x) = b_1 + b_2 x + \cdots + b_n x^{n-1} \leftrightarrow b = (b_1, b_2, \ldots, b_n)^T \]

then
\[ \alpha p(x) = \alpha a_1 + \alpha a_2 x + \cdots + \alpha a_n x^{n-1} \]
\[ \alpha a = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)^T \]

and
\[ (p + q)(x) = (a_1 + b_1) + (a_2 + b_2)x + \cdots + (a_n + b_n)x^{n-1} \]
\[ a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)^T \]

Thus
\[ \alpha p \leftrightarrow \alpha a \quad \text{and} \quad p + q \leftrightarrow a + b \]
SECTION 2

7. $C^n[a,b]$ is a nonempty subset of $C[a,b]$. If $f \in C^n[a,b]$, then $f^{(n)}$ is continuous. Any scalar multiple of a continuous function is continuous. Thus for any scalar $\alpha$, the function

$$(\alpha f)^{(n)} = \alpha f^{(n)}$$

is also continuous and hence $\alpha f \in C^n[a,b]$. If $f$ and $g$ are vectors in $C^n[a,b]$ then both have continuous $n$th derivatives and their sum will also have a continuous $n$th derivative. Thus $f + g \in C^n[a,b]$ and therefore $C^n[a,b]$ is a subspace of $C[a,b]$.

8. (a) If $B \in S_1$, then $AB = BA$. It follows that

$$A(\alpha B) = \alpha AB = \alpha BA = (\alpha B)A$$

and hence $\alpha B \in S_1$.

If $B$ and $C$ are in $S_1$, then

$$AB = BA \quad \text{and} \quad AC = CA$$

thus

$$A(B + C) = AB + AC = BA + CA = (B + C)A$$

and hence $B + C \in S_1$. Therefore $S_1$ is a subspace of $R^{2 \times 2}$.

(b) If $B \in S_2$, then $AB \neq BA$. However, for the scalar $0$, we have

$$0B = O \notin S_2$$

Therefore $S_2$ is not a subspace. (Also, $S_2$ is not closed under addition.)

(c) If $B \in S_2$, then $BA = O$. It follows that

$$(\alpha B)A = \alpha (BA) = \alpha O = O$$

Therefore, $\alpha B \in S_2$. If $B$ and $C$ are in $S_2$, then

$$BA = O \quad \text{and} \quad CA = O$$

It follows that

$$(B + C)A = BA + CA = O + O = O$$

Therefore $B + C \in S_2$ and hence $S_2$ is a subspace of $R^{2 \times 2}$.

11 (a) $x \in \text{Span}(x_1, x_2)$ if and only if there exist scalars $c_1$ and $c_2$ such that

$$c_1x_1 + c_2x_2 = x$$

Thus $x \in \text{Span}(x_1, x_2)$ if and only if the system $Xc = x$ is consistent. To determine whether or not the system is consistent we can compute the row echelon form of the augmented matrix $(X \vert x)$.

$$
\begin{pmatrix}
-1 & 3 & 2 \\
2 & 4 & 6 \\
3 & 2 & 6
\end{pmatrix}
\begin{pmatrix}
1 & -3 & -2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

The system is inconsistent and therefore $x \notin \text{Span}(x_1, x_2)$. 

(c)\
\[
\begin{pmatrix}
-1 & 3 \\
2 & 4 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
-9 \\
-2 \\
5
\end{pmatrix}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -3 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
-2 \\
-2 \\
0
\end{pmatrix}
\]

The system is consistent and therefore \( y \in \text{Span}(x_1, x_2) \).

12. If \( x_k \notin \text{Span}(x_1, x_2, \ldots, x_{k-1}) \), then \( \{x_1, x_2, \ldots, x_{k-1}\} \) cannot be a spanning set. On the other hand if \( x_k \in \text{Span}(x_1, x_2, \ldots, x_{k-1}) \), then
\[
\text{Span}(x_1, x_2, \ldots, x_k) = \text{Span}(x_1, x_2, \ldots, x_{k-1})
\]
and hence the \( k - 1 \) vectors will span the entire vector space.

13. If \( A = (a_{ij}) \) is any element of \( R^{2 \times 2} \), then
\[
A = \begin{pmatrix}
a_{11} & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & a_{12} \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
a_{21} & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & a_{22}
\end{pmatrix}
\]

\[
= a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}
\]

15. If \( \{a_n\} \in S_0 \), then \( a_n \to 0 \) as \( n \to \infty \). If \( \alpha \) is any scalar, then \( \alpha a_n \to 0 \) as \( n \to \infty \) and hence \( \{\alpha a_n\} \in S_0 \). If \( \{b_n\} \) is also an element of \( S_0 \), then \( b_n \to 0 \) as \( n \to \infty \) and it follows that
\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = 0 + 0 = 0
\]

Therefore \( \{a_n + b_n\} \in S_0 \), and it follows that \( S_0 \) is a subspace of \( S \).

16. Let \( S \neq \{0\} \) be a subspace of \( R^1 \) and let \( a \) be an arbitrary element of \( R^1 \). If \( s \) is a nonzero element of \( S \), then we can define a scalar \( \alpha \) to be the real number \( a/s \). Since \( S \) is a subspace it follows that
\[
\alpha s = \frac{a}{s} s = a
\]
is an element of \( S \). Therefore \( S = R^1 \).

17. (a) implies (b).

If \( N(A) = \{0\} \), then \( Ax = 0 \) has only the trivial solution \( x = 0 \). By Theorem 1.4.3, \( A \) must be nonsingular.

(b) implies (c).

If \( A \) is nonsingular then \( Ax = b \) if and only if \( x = A^{-1}b \). Thus \( A^{-1}b \) is the unique solution to \( Ax = b \).

(c) implies (a).

If the equation \( Ax = b \) has a unique solution for each \( b \), then in particular for \( b = 0 \) the solution \( x = 0 \) must be unique. Therefore \( N(A) = \{0\} \).

18. Let \( \alpha \) be a scalar and let \( x \) and \( y \) be elements of \( U \cap V \). The vectors \( x \) and \( y \) are elements of both \( U \) and \( V \). Since \( U \) and \( V \) are subspaces it follows that
\[
\alpha x \in U \quad \text{and} \quad x + y \in U
\]
\[\alpha x \in V \quad \text{and} \quad x + y \in V\]

Therefore
\[\alpha x \in U \cap V \quad \text{and} \quad x + y \in U \cap V\]

Thus \(U \cap V\) is a subspace of \(W\).

19. \(S \cup T\) is not a subspace of \(R^2\).

\[S \cup T = \{(s, t)^T | s = 0 \text{ or } t = 0\}\]

The vectors \(e_1\) and \(e_2\) are both in \(S \cup T\), however, \(e_1 + e_2 \not\in S \cup T\).

20. If \(z \in U + V\), then \(z = u + v\) where \(u \in U\) and \(v \in V\). Since \(U\) and \(V\) are subspaces it follows that

\[\alpha u \in U \quad \text{and} \quad \alpha v \in V\]

for all scalars \(\alpha\). Thus

\[\alpha z = \alpha u + \alpha v\]

is an element of \(U + V\). If \(z_1\) and \(z_2\) are elements of \(U + V\), then

\[z_1 = u_1 + v_1 \quad \text{and} \quad z_2 = u_2 + v_2\]

where \(u_1, u_2 \in U\) and \(v_1, v_2 \in V\). Since \(U\) and \(V\) are subspaces it follows that

\[u_1 + u_2 \in U \quad \text{and} \quad v_1 + v_2 \in V\]

Thus

\[z_1 + z_2 = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2)\]

is an element of \(U + V\). Therefore \(U + V\) is a subspace of \(W\).

\section*{SECTION 3}

5. (a) If \(x_{k+1} \in \text{Span}(x_1, x_2, \ldots, x_k)\), then the new set of vectors will be linearly dependent. To see this suppose that

\[x_{k+1} = c_1x_1 + c_2x_2 + \cdots + c_kx_k\]

If we set \(c_{k+1} = -1\), then

\[c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_{k+1}x_{k+1} = 0\]

with at least one of the coefficients, namely \(c_{k+1}\), being nonzero.

On the other hand if \(x_{k+1} \not\in \text{Span}(x_1, x_2, \ldots, x_k)\) and

\[c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_{k+1}x_{k+1} = 0\]

then \(c_{k+1} = 0\) (otherwise we could solve for \(x_{k+1}\) in terms of the other vectors). But then

\[c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_kx_k = 0\]

and it follows from the independence of \(x_1, \ldots, x_k\) that all of the \(c_i\) coefficients are zero and hence that \(x_1, \ldots, x_{k+1}\) are linearly independent. Thus if \(x_1, \ldots, x_k\) are linearly independent and we add a vector \(x_{k+1}\) to
the collection, then the new set of vectors will be linearly independent if and only if \( x_{k+1} \not\in \text{Span}(x_1, x_2, \ldots, x_k) \)

(b) Suppose that \( x_1, x_2, \ldots, x_k \) are linearly independent. To test whether or not \( x_1, x_2, \ldots, x_{k-1} \) are linearly independent consider the equation

\[
c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = 0
\]

If \( c_1, c_2, \ldots, c_{k-1} \) work in equation (1), then

\[
c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + 0x_k = 0
\]

and it follows from the independence of \( x_1, \ldots, x_k \) that

\[
c_1 = c_2 = \cdots = c_{k-1} = 0
\]

and hence \( x_1, \ldots, x_{k-1} \) must be linearly independent.

7. (a) \( W(\cos n\pi, \sin n\pi) = \pi \). Since the Wronskian is not identically zero the vectors are linearly independent.

(b) \( W(x, e^x, e^{2x}) = 2(x - 1)e^{3x} \neq 0 \)

(c) \( W(x^2, \ln(1 + x^2), 1 + x^2) = \frac{-8x^3}{(1 + x^2)^2} \neq 0 \)

(d) To see that \( x^3 \) and \( |x|^3 \) are linearly independent suppose

\[
c_1x^3 + c_2|x|^3 = 0
\]

on \([-1, 1]\). Setting \( x = 1 \) and \( x = -1 \) we get

\[
c_1 + c_2 = 0
\]

\[
-c_1 + c_2 = 0
\]

The only solution to this system is \( c_1 = c_2 = 0 \). Thus \( x^3 \) and \( |x|^3 \) are linearly independent.

8. The vectors are linearly dependent since

\[\cos x - 1 + 2\sin^2 \frac{x}{2} = 0\]

on \([-\pi, \pi]\).

10. (a) If

\[
c_1(2x) + c_2|x| = 0
\]

for all \( x \) in \([-1, 1]\), then in particular we have

\[
-2c_1 + c_2 = 0 \quad (x = -1)
\]

\[
2c_1 + c_2 = 0 \quad (x = 1)
\]

and hence \( c_1 = c_2 = 0 \). Therefore \( 2x \) and \( |x| \) are linearly independent in \( C[-1, 1] \).

(b) For all \( x \) in \([0, 1]\)

\[
1 \cdot 2x + (-2)|x| = 0
\]

Therefore \( 2x \) and \( |x| \) are linearly dependent in \( C[0, 1] \).
11. Let \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) be vectors in a vector space \( V \). If one of the vectors, say \( \mathbf{v}_1 \), is the zero vector then set

\[
c_1 = 1, \quad c_2 = c_3 = \cdots = c_n = 0
\]

Since

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0}
\]

and \( c_1 \neq 0 \), it follows that \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) are linearly dependent.

12. If \( \mathbf{v}_1 = \alpha \mathbf{v}_2 \), then

\[
1 \mathbf{v}_1 - \alpha \mathbf{v}_2 = \mathbf{0}
\]

and hence \( \mathbf{v}_1, \mathbf{v}_2 \) are linearly dependent. Conversely, if \( \mathbf{v}_1, \mathbf{v}_2 \) are linearly dependent, then there exists scalars \( c_1, c_2 \), not both zero, such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0}
\]

If say \( c_1 \neq 0 \), then

\[
\mathbf{v}_1 = -\frac{c_2}{c_1} \mathbf{v}_2
\]

13. Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) be a linearly independent set of vectors and suppose there is a subset, say \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) of linearly dependent vectors. This would imply that there exist scalars \( c_1, c_2, \ldots, c_k \), not all zero, such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \mathbf{0}
\]

but then

\[
c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}
\]

This contradicts the original assumption that \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are linearly independent.

14. If \( \mathbf{x} \in N(A) \) then \( A\mathbf{x} = \mathbf{0} \). Partitioning \( A \) into columns and \( \mathbf{x} \) into rows and performing the block multiplication, we get

\[
x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{0}
\]

Since \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \) are linearly independent it follows that

\[
x_1 = x_2 = \cdots = x_n = 0
\]

Therefore \( \mathbf{x} = \mathbf{0} \) and hence \( N(A) = \{0\} \).

15. If

\[
c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \cdots + c_k \mathbf{y}_k = \mathbf{0}
\]

then

\[
c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 + \cdots + c_k A\mathbf{x}_k = \mathbf{0}
\]

\[
A(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_k \mathbf{x}_k) = \mathbf{0}
\]

Since \( A \) is nonsingular it follows that

\[
c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_k \mathbf{x}_k = \mathbf{0}
\]

and since \( \mathbf{x}_1, \ldots, \mathbf{x}_k \) are linearly independent it follows that

\[
c_1 = c_2 = \cdots = c_k = 0
\]