SEQUENCES

A sequence is a list of numbers in a definite order

\[ a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \]

Each number in the sequence is called a "term" and is referred to by its subscript, i.e.

\[ a_1 \text{ is the "first term" of the sequence, } a_2 \text{ is the "second term", etc, so } a_{1734} \text{ is the "1,734th term" of the sequence.} \]

We have several notations to indicate we are thinking of a sequence:

- \( \{ a_1, a_2, a_3, \ldots \} \) or \( \{ a_n \} \), or \( \{ a_n \}_{n=1}^{\infty} \) all mean the same thing
Examples: One way to define a sequence is to give a formula for the \( n^{th} \)-term.

(i) \( \left\{ \frac{n^2}{n^2+1} \right\}_{n=1}^{\infty} \), or \( a_n = \frac{n^2}{n^2+1} \),

or \( \left\{ \frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \ldots, \frac{n^2}{n^2+1}, \ldots \right\} \)

(ii) \( \left\{ \frac{(-1)^n \sqrt{n+1}}{2n+1} \right\}_{n=1}^{\infty} \), or \( a_n = \frac{(-1)^n \sqrt{n+1}}{2n+1} \),

or \( \left\{ \frac{-\sqrt{2}}{4}, \frac{\sqrt{3}}{8}, \frac{-\sqrt{4}}{9}, \ldots, \frac{(-1)^n \sqrt{n+1}}{2n+1}, \ldots \right\} \)

(iii) \( \{ \sin \left( \frac{n\pi}{3} \right) \}_{n=0}^{\infty} \), or, \( a_n = \sin \left( \frac{n\pi}{3} \right) \), \( n \geq 0 \)

or, \( \left\{ 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \ldots \right\} \)

Note: We can think of a sequence as a function on the integers.
So we can graph the sequence:

Ex: \( a_n = \frac{1}{n} \), \( n \geq 1 \)

Another way to define a sequence is recursively, that is give away to compute the next term from the one you know. Ex: Fibonacci Sequence:

\( f_1 = 1, \ f_2 = 1, \) and \( f_{n+2} = f_n + f_{n+1} \)

for \( n \geq 2 \)

\( \{1, 1, 2, 3, 5, 8, 13, 21, 34, 45 \ldots \} \)
We can talk about the limit of a sequence: For example \( \lim_{n \to \infty} \frac{n^2}{n^2-1} \) 

Or graph the sequence along the line 

Note that 
\[
\frac{n^2}{n^2-1} - 1 = \frac{n^2 - (n^2-1)}{n^2-1} = \frac{1}{n^2-1}
\]

This says we can make \( \frac{n^2}{n^2-1} \) as close to 1 as we like, provided we let \( n \) get large enough, (and for all subsequent \( n \) as well).
So we say \( \lim_{n \to \infty} \frac{n^2}{n^2 - 1} = 1 \).

**Definition:** The sequence \( \{a_n\} \) has limit \( L \) if for any \( \varepsilon > 0 \) there is an \( N = N_\varepsilon \) so that if \( n > N \),

\[ |a_n - L| < \varepsilon. \]

We write \( \lim_{n \to \infty} a_n = L \), or \( (a_n \to L) \) as \( n \to \infty \).

This means eventually, for any \( \varepsilon > 0 \), the sequence has all its terms in the interval \( (L - \varepsilon, L + \varepsilon) \).
Note how this is similar to the definition $\lim_{x \to \infty} f(x) = L$.

**Theorem:** If $\lim_{x \to \infty} f(x) = L$ and $a_n = f(n)$, then $\lim_{n \to \infty} a_n = L$.

**Ex:** $\lim_{x \to \infty} \frac{1}{x} = 0$, so if $a_n = \frac{1}{n}$, $n \geq 1$, then $\lim_{n \to \infty} a_n = 0$

**IF** $\lim_{n \to \infty} a_n = L$, i.e., if the limit exists, then, we say the sequence converges (or is convergent). Otherwise the sequence diverges.

We say $\lim_{n \to \infty} a_n = \infty$ if for any $M > 0$ there is an $N$ so that $a_n > M$ for all $n > N$. 
Note: If \( \lim_{n \to \infty} a_n = 0 \) then the sequence is DIVERGENT!

There are limit laws for sequences similar to those for limits of functions:

\[ \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \]

Provided both limits exist on the right hand side of the equation.

Note: These laws are given on page 694.
SQUEEZE THM APPLIES TO SEQUENCES

If \( a_n \leq b_n \leq c_n \) for all \( n \) and \( \lim \limits_{n \to \infty} a_n = L = \lim \limits_{n \to \infty} c_n \), then

\[ \lim \limits_{n \to \infty} b_n = L. \]

IMPORTANT FACT:
If \( \lim \limits_{n \to \infty} |a_n| = 0 \) then \( \lim \limits_{n \to \infty} a_n = 0 \)

Ex: Find \( \lim \limits_{n \to \infty} \frac{n^2}{n^2-1} \)

\[ \lim \limits_{n \to \infty} \frac{n^2}{n^2-1} = \lim \limits_{n \to \infty} \left( \frac{1}{1-\frac{1}{n^2}} \right) \]

\[ \frac{1}{1-0} = \frac{1}{1+0} = 1 \]
\[
\lim_{{n \to \infty}} \frac{n}{{1 + \sqrt{n}}} = ?
\]

\[
\frac{n}{{1 + \sqrt{n}}} = \frac{n}{{1 + n^{\frac{1}{2}}} = \frac{n^{\frac{1}{2}} \cdot n^{\frac{1}{2}}}{{n^{\frac{1}{2}} \cdot (\frac{1}{n^{\frac{1}{2}}} + 1)}}
\]

\[
\lim_{{n \to \infty}} \frac{n}{{1 + \sqrt{n}}} = \lim_{{n \to \infty}} \frac{\sqrt{n}}{{\frac{1}{\sqrt{n}} + 1}}
\]

\[
= \frac{\lim_{{n \to \infty}} \sqrt{n}}{{\lim_{{n \to \infty}} \frac{1}{\sqrt{n}} + 1}
\]

\[
(\lim_{{n \to \infty}} \sqrt{n} = \infty)
\]

So \(\frac{n}{{1 + \sqrt{n}}}\) is divergent.
\[
\lim_{n \to \infty} \frac{n!}{n^n}
\]

\[n! = 1 \cdot 2 \cdot 3 \cdots n\]

\[n^n = \underbrace{n \cdot n \cdots n}_{n \text{-factors}}\]

\[a_1 = \frac{1!}{1^1} = \frac{1}{1} = 1\]

\[a_2 = \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{2}\]

\[a_3 = \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3} = \frac{2}{9}\]

\[\vdots\]

\[a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \]

\[
\frac{1}{n} \left( \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n-1}{n} \right) \leq \frac{1}{n} \cdot (1 \cdot 1 \cdots 1)
\]

Note:
\[0 \leq a_n \leq \frac{1}{n}\]
Since \( \lim_{n \to \infty} 0 = 0 = \lim_{n \to \infty} \frac{1}{n} \),

\[
\lim_{n \to \infty} \frac{n!}{n^n} = 0 \text{ by Squeeze Theorem}.
\]

How about \( a_n = r^n \) for some fixed \( r \)?

\[
\lim_{n \to \infty} r^n = \begin{cases} 
\infty & \text{if } |r| > 1 \text{ or DNE} \\
1 & \text{if } r = 1 \\
0 & \text{if } |r| < 1
\end{cases}
\]

Does not exist if \( r = -1 \).

**Ex:** \( \left( \frac{1}{2} \right)^n \to 0 \) as \( n \to \infty \).

Since \( \left( \frac{4}{5} \right)^n = \frac{1}{2^n} \) can be made as small as possible.

Note \( a_n = (-1)^n \)

\( \{1, -1, 1, -1, 1, -1, \ldots \} \) has no limit.
A sequence is **increasing** if \( a_n > a_{n-1} \) for all \( n \). A sequence is **decreasing** if \( a_n \leq a_{n-1} \) for all \( n \).

Ex: \( \frac{1}{n} \) is a decreasing sequence

\[ \frac{n}{n+1} \] is an increasing sequence

Since

\[
\frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - (n+1)(n-1)}{(n+1)n}
\]

\[
= \frac{n^2 - (n^2 - 1)}{(n+1)n} = \frac{1}{(n+1)n} > 0 \] i.e.

\[
\frac{n}{n+1} > \frac{n-1}{n}.
\]

A sequence is **bounded above** if there is some \( M \) with \( a_n \leq M \) for all \( n \).
**Ex:**

Since \( \frac{n}{n+1} < 1 \),

\( \{ \frac{n}{n+1} \} \) is bounded above.

A sequence is **bounded below** if there is some \( M \) with \( M \leq a_n \) for all \( n \);

**Ex:** \( \frac{1}{n} \) is bounded below since \( \frac{1}{n} > 0 \) for all \( n \geq 1 \).

Every bounded monotonic sequence converges:

**Bounded** = bounded above and below

**Monotonic** = either increasing or decreasing
Use differentiation: For increasing/decreasing, limits (L'Hopital's rule)

\[ a_n = \frac{n}{2^n}, \quad n > 2 \]

Consider \( f(x) = \frac{x}{2^x} \).

\[ f'(x) = \frac{2^x \cdot 1 - (x \ln 2)2^x}{2^{2x}} \quad \text{(Quotient rule)} \]

\[ = \frac{1-x \ln 2}{2^x} < 0 \quad \text{if} \quad x \ln 2 > 1 \quad \text{i.e.} \]

\[ x > \frac{1}{\ln 2} > 2. \]

So \( f \) is decreasing \( \Rightarrow \)

\( a_n \) is decreasing.

\( a_n \) is bounded, below by 0, above by 1, so \( a_n \) is convergent.
Ex: \( \frac{\ln(n^2)}{n} = a_n \)

Is \( \{ a_n \} \) convergent?

\[ \lim_{n \to \infty} a_n = ? \]

If \( f(x) = \frac{\ln(x^2)}{x} \), to find

\[ \lim_{x \to \infty} f(x) \]
we use L'Hopital's rule:

\[ \lim_{x \to \infty} \frac{\ln(x^2)}{x} = \lim_{x \to \infty} \frac{1}{x^2} \cdot 2x \]

\[ = \lim_{x \to \infty} \frac{2}{x} = 0. \]

So \( \lim_{n \to \infty} \frac{\ln(n^2)}{n} = 0. \)
SERIES

If \( \{a_n\} \) is a sequence we can ask if there is any way to make sense of
\[
a_1 + a_2 + a_3 + \ldots + a_n + \ldots
\]

Intuition: You shouldn't be able to sum up infinitely many terms.

On the other hand, remember improper integrals like
\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx.
\]
We thought this couldn't make sense either.

We will say more about the connection between improper integrals and series next class.
Consider the "sum"
\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots \]

Think of a street of length 1 mile. Every minute you walk halfway from the current position to the end.

So it seems that
\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1 \]

\( a_i = \frac{1}{2^n} \) is our sequence.

Let \( S_n = a_1 + a_2 + \ldots + a_n \)

This is a partial sum of the sequence
3.

\[ S_n = \text{\textit{n}th Partial Sum.} \]

For our sequence

\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{3}{4} \]
\[ S_3 = \frac{7}{8} \]
\[ S_4 = \frac{15}{14} \]
\[ S_5 = \frac{31}{32} \]
\[ S_6 = \frac{63}{64} \]
\[ \vdots \]
\[ S_n = \frac{2^n - 1}{2^n} \]

So the sequence of partial sums \( \{S_n\} \) converges to 1.

Definition: If \( \{a_n\} \) is a sequence, write \( \sum a_n \) or \( \sum_{n=1}^{\infty} a_n \) for the series \( a_1 + a_2 + \ldots + a_n + \ldots \)

(as a formal expression)
**Def:** Given a series \( \sum_{n=1}^{\infty} a_n \), let 
\( S_n \) denote its \( n \)th partial sum,
\[
S_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k
\]
If the sequence \( \{S_n\} \) is convergent, and if \( \lim_{n \to \infty} S_n = s \) we say \( \sum a_n \) is convergent and write
\[
a_1 + a_2 + \ldots + a_n + \ldots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s
\]
If \( \lim_{n \to \infty} S_n \) does not exist, then the series is called **divergent**
Example: Let $r$ be a fixed positive number, and $a = 1/n$ a constant.

Consider the series

$$a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$$

Assume $r \neq 1$, since $a + ar + \ldots + a$ will diverge.

Consider the partial sum $S_n$:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad (1)$$

Note

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad (2)$$

Subtract (1) from (2):

$$(r-1)S_n = ar^n - a = a(r^n - 1)$$

So

$$S_n = \frac{a(r^n - 1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$$

So $S_n$ converges if and only if

$$\lim_{n \to \infty} r^n$$ exists
Recall (Sec. 11.1) \( r^n \to 0 \) if \( |r| < 1 \) and \( \{r^n\} \) is divergent otherwise (for \( r \neq 1 \)).

Thus:

The geometric series:

\[
(a \neq 0) \quad \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots = \frac{a}{1-r} + ar^n + \ldots
\]

is convergent if \( |r| < 1 \) and then

\[
\sum_{n=1}^{\infty} ar^{n-1} = a \left( \frac{1}{1-r} \right) = \frac{a}{1-r}.
\]

If \( |r| \geq 1 \) The geometric series diverges.

EX: \( a = 1, \ r = \frac{1}{2} \)

\[
\sum_{n=1}^{9} \left( \frac{1}{2} \right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2
\]
Consider
\[ 4 - \frac{8}{5} + \frac{16}{25} - \frac{32}{125} + \frac{64}{625} - \cdots \]

This is a geometric series with
\[ a = 4 \text{ (first term)} \text{ and} \]
\[ r = \left(\frac{-\frac{8}{5}}{4}\right) = -\frac{2}{5} \]

Note:
\[ \left(\frac{\frac{16}{25}}{-\frac{8}{5}}\right) = -\frac{2}{5} = \frac{-\frac{32}{125}}{-\frac{64}{625}} = \cdots \]

Our series is
\[ \sum_{n=1}^{8} 4 \cdot \left(-\frac{2}{5}\right)^{n-1} \]

so we get
\[ \sum_{n=1}^{8} 4 \left(-\frac{2}{5}\right)^{n-1} = \frac{4}{1 - \left(-\frac{2}{5}\right)} = \frac{4}{\left(\frac{3}{5}\right)} = \frac{20}{3} \]
TElescoping Series:

\[ \sum_{n=1}^{\infty} \frac{1}{2n(n+1)} \]

By partial fractions:

\[ \frac{1}{2n(n+1)} = \frac{1}{2n} - \frac{1}{2(n+1)} \]

So our partial sums are

\[ S_n = \frac{1}{2 \cdot 1 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2n(n+1)} = \]

\[ \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{8} \right) + \ldots + \left( \frac{1}{2(n-1)} - \frac{1}{2n} \right) + \left( \frac{1}{2n} - \frac{1}{2(n+1)} \right) \]

\[ = \frac{1}{2} - \frac{1}{2(n+1)} \]
9. Since:
\[ S_n \to \frac{1}{2} \quad \text{as} \quad n \to \infty, \]
\[
\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2}.
\]

Harmonic Series:

The series \[ \sum_{n=1}^{\infty} \frac{1}{n} \] diverges:

Why?

\[ S_1 = 1 \]
\[ S_2 = 1 + \frac{1}{2} \]
\[ S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \]

Since \[ \frac{1}{3} > \frac{1}{4}, \quad \frac{1}{3} + \frac{1}{4} > \frac{3}{4} = \frac{1}{2}, \quad \text{so} \]
\[ S_4 > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \]

\[ S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \]

Since \[ \frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8}, \quad \text{so} \]
\[ S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \]
\[ S_{10} = 1 + \frac{1}{2} + \cdots + \frac{1}{10} + \left( \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} \right) \]

\[ > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} \]

For any \( n \):

\[ S_{2n} = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \cdots + \left( \frac{1}{2^{n-1} + 1} + \frac{1}{2^{n-2} + 1} + \cdots + \frac{1}{2^n} \right) \]

\[ > 1 + \frac{n}{2} \]

So \( \lim_{n \to \infty} S_n = \infty \), does not converge.
**IMPORTANT FACT**

If \( \sum_{n=1}^{\infty} a_n \) is convergent

then \( \lim_{n \to \infty} a_n = 0 \)

**NOTE** REVERSING "IF" AND "THEN"

DOES NOT WORK.

Ex: \( \frac{1}{n} \to 0 \)

but \( \sum \frac{1}{n} \) diverges
Divergence Test: If \( \lim_{n \to \infty} a_n \) does not exist, or \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

Example:

\[
\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}
\]

\[
a_n = \frac{(n+1)^2}{n(n+2)} = \frac{n^2 + 2n + 1}{n^2 + 2n}
= \frac{n^2}{n^2} \left( \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}} \right)
\]

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}} \right)
= \lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{2}{n} + \lim_{n \to \infty} \frac{1}{n^2}
= \frac{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{2}{n} + \lim_{n \to \infty} \frac{1}{n^2}}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{2}{n}}
= \frac{1 + 0 + 0}{1 + 0} = 1
\]

So, \( \sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)} \) diverges.
Rules for Infinite Series:

Suppose $\sum an$ and $\sum bn$ are both convergent.

Then, for any constant $C$, 

$\sum C an$, $\sum (an+bn)$, and 

$\sum (an-bn)$ are all convergent.

Furthermore:

\[ \sum_{n=1}^{\infty} can = c \sum_{n=1}^{\infty} an, \]

\[ \sum_{n=1}^{\infty} (an+bn) = \sum_{n=1}^{\infty} an + \sum_{n=1}^{\infty} bn, \]

\[ \sum_{n=1}^{\infty} (an-bn) = \sum_{n=1}^{\infty} an - \sum_{n=1}^{\infty} bn. \]
Use these rules to find

\[ \sum_{n=1}^{8} \left( \frac{1}{n(n+1)} - \frac{4}{3^n} \right) \]

We saw \[ \sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2} \]

So \[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{2n(n+1)} \right) \frac{1}{2} = (\frac{1}{2}) \]

= 2 \left( \frac{1}{2} \right) = 1

Also \[ \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \ldots \]

= \frac{(-1+1)}{1-\frac{1}{3}} = -1 + \frac{3}{2} = \frac{1}{2}

So \[ \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} - \frac{4}{3^n} \right) = \]

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - 4 \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2} - 4 \left( \frac{1}{2} \right) = -\frac{3}{2} \]