Approximation by ridge functions with weights in a specified set

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Extreme Learning Machines

Extreme Learning Machines are neural networks with one hidden layer where the training is only carried out on the weights in the *output*. The weights before the activation/sigmoidal functions are generated randomly are left unchanged.

Approximation by ridge functions

We consider approximating functions $f(\boldsymbol{x})$ for $\boldsymbol{x} \in [-1, +1]^m = J^m$ by functions in

$$V_{\mathcal{W}} := \text{span}\left\{ \boldsymbol{x} \mapsto \varphi(\boldsymbol{w}^T \boldsymbol{x}) \mid \varphi \in C(\mathbb{R}),\ \boldsymbol{w} \in \mathcal{W} \right\}$$

for $\mathcal{W}$ a specific subset in $\mathbb{R}^m$.

Questions:

- For what sets $\mathcal{W}$ is $V_{\mathcal{W}}$ dense in $C(J^m)$?
- If $V_{\mathcal{W}}$ is not dense in $C(J^m)$, how well can we approximate (nice) functions by functions in $V_{\mathcal{W}}$?

General framework

We work in a Banach space X; the approximating functions form a subspace $V \subset X$.

If $\overline{V} = X$ then every object in X can be approximated (arbitrarily well) by elements of V.

But if $\overline{V} \neq X$, then for every $\epsilon > 0$ there are functions $0 \neq f \in X$ where $\inf_{g \in V} \|f - g\|_X \geq (1 - \epsilon) \|f\|_X$. So

$$\sup_{f : \|f\|_X = 1} \inf_{g \in V} \|f - g\|_X \qquad \text{is either 0 or 1.}$$

Choose a Banach subspace Z compactly embedded in X and we look determine

$$m(V; Z, X) = \sup_{f:\|f\|_Z=1} \inf_{g\in V} \|f - g\|_X .$$

In our case we use:

- $X = C(J^m)$
- $V = V_{\mathcal{W}}$
- $Z = \mathrm{Lip}(J^m)$, the space of Lipschitz functions with semi-norm
$$|f|_Z = \sup_{\boldsymbol{x},\boldsymbol{y}\in J^m} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|_2}$$
(We can quotient out constant functions since $V_{\mathcal{W}}$ always contains these.)

We say $f \in X$ is *unapproximable* by V if

$$\|f\|_X \leq \|f - g\|_X \qquad \text{for all } g \in V.$$

For any $f \in X$ if $h \in V$ is the closest point in V to f then $f - h$ is unapproximable by V.

The existence of a closest point is assured if X is a reflexive Banach space, but generally false otherwise.

Separating Hyperplane Theorem

Theorem: If $C \subset X$ is closed and convex and $y \notin C$, then there is a $\mu \in X'$ and $b \in \mathbb{R}$ where

$$\begin{aligned} \langle y, \mu \rangle + b &> 0 \\ \langle z, \mu \rangle + b &\leq 0 \qquad \text{for all } z \in C \end{aligned}$$

Specifically, if C is a closed subspace of X, then ν satisfies

$$\begin{aligned} \langle y, \mu \rangle &> 0 \\ \langle z, \mu \rangle &= 0 \qquad \text{for all } z \in C. \end{aligned}$$

Cybenko's universal approximation result

George Cybenko's paper from 1989 shows that if (for example) $\sigma(u) = \tanh(u)$ then

$$\overline{\operatorname{span}\left\{\, \boldsymbol{x} \mapsto \sigma(\boldsymbol{w}^T\boldsymbol{x} + b) \mid \boldsymbol{w} \in \mathbb{R}^m,\ b \in \mathbb{R} \,\right\}} = C(J^m).$$

The proof uses the Separating Hyperplane Theorem

Note: $C(J^m)' = \mathcal{M}(J^m)$, the space of signed Borel measures on J^m with bounded variation and $\langle g, \mu \rangle = \int g(\boldsymbol{x})\, d\mu(\boldsymbol{x})$.

For the μ in the Separating Hyperplane Theorem

$$\int \sigma(a\boldsymbol{w}^T\boldsymbol{x} + b)\, d\mu(\boldsymbol{x}) = 0 \qquad \text{for all } a, b \in \mathbb{R}$$

so we can show that

$$0 = \mu^{\boldsymbol{w}}(F) := \mu\left(\left\{\, \boldsymbol{x} \mid \boldsymbol{w}^T\boldsymbol{x} \in F \,\right\}\right) \qquad \text{for all Borel } F \subset \mathbb{R}.$$

Fourier Transforms

A Borel measure μ with bounded variation has a Fourier Transform

$$\widehat{\mu}(\boldsymbol{\xi}) = \int_{\mathbb{R}^m} e^{-i\boldsymbol{\xi}^T\boldsymbol{x}}\, d\mu(\boldsymbol{x}).$$

Note that for the μ from the SHT

$$\begin{aligned}\widehat{\mu}(s\boldsymbol{w}) &= \int_{\mathbb{R}^m} e^{-is\boldsymbol{w}^T\boldsymbol{x}}\, d\mu(\boldsymbol{x}) \\ &= \int_{\mathbb{R}} e^{-ist}\, d\mu^{\boldsymbol{w}}(t) = \widehat{\mu^{\boldsymbol{w}}}(s) = 0.\end{aligned}$$

For Cybenko's result, this is true for all $\boldsymbol{w} \in \mathbb{R}^m$ so $\widehat{\mu}(\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi}$, and so $\mu = 0$ contradicting the SHT.

Thus there is no f in $C(J^m)$ that is not in

$$\overline{\operatorname{span}\left\{\, \boldsymbol{x} \mapsto \sigma(\boldsymbol{w}^T\boldsymbol{x} + b) \mid \boldsymbol{w} \in \mathbb{R}^m,\ b \in \mathbb{R} \,\right\}}$$

What about specific (finite) $\mathcal{W}$?

What about spans of ridge functions

$$V_{\mathcal{W}} := \operatorname{span}\left\{ \boldsymbol{x} \mapsto \varphi(\boldsymbol{w}^T \boldsymbol{x}) \mid \varphi \in C(\mathbb{R}),\ \boldsymbol{w} \in \mathcal{W} \right\} ?$$

We can get *lower bounds* on how badly a Lipschitz function f can be approximated by $V_{\mathcal{W}}$ as follows: Pick a measure μ with support in J^m where $\mu^{\boldsymbol{w}} = 0$ for every $\boldsymbol{w} \in \mathcal{W}$. Then look for a function f where $\langle f, \mu \rangle = \|f\|_\infty \|\mu\|_{\mathcal{M}} \neq 0$.

The measure μ has to satisfy $\widehat{\mu}(t\boldsymbol{w}) = 0$ for all $t \in \mathbb{R}$ and $\boldsymbol{w} \in \mathcal{W}$.

Example: $\mathcal{W} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

The Fourier transform $\widehat{\mu}(t\boldsymbol{e}_1) = \widehat{\mu}(t\boldsymbol{e}_2) = 0$ for all $t \in \mathbb{R}$.

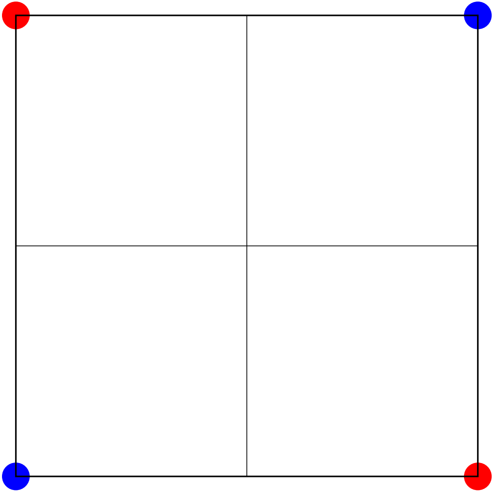
Since the Fourier transform of $\delta_{\boldsymbol{v}} = \delta(\cdot - \boldsymbol{v})$ is $\widehat{\delta_{\boldsymbol{v}}}(\boldsymbol{\xi}) = \exp(-i\boldsymbol{\xi}^T\boldsymbol{v})$, so we look for $\widehat{\mu}(\boldsymbol{\xi})$ that involves complex exponentials $\exp(-i\boldsymbol{\xi}^T\boldsymbol{v})$ for $\boldsymbol{v} \in J^m$.

Note: $\widehat{\mu}(\boldsymbol{\xi})$ is complex analytic everywhere (entire) so we can look for Taylor series. So...

$$\widehat{\mu}(\boldsymbol{\xi}) = c\,\xi_1\,\xi_2 + \cdots$$

We can put $\widehat{\mu}(\boldsymbol{\xi}) = \left(e^{-i\xi_1} - e^{+i\xi_1}\right)\left(e^{-i\xi_2} - e^{+i\xi_2}\right)$, $(c = (-2i)^2)$.

Here μ is a sum of δ-functions at (x_1, x_2) with each $x_i = \pm 1$, and the weights at each of these points is $x_1 x_2$.



$= \operatorname{supp} \mu_-$, $= \operatorname{supp} \mu_+$

Now we need to find a function f with, say, $\|f\|_\infty = 1$ and $\langle f, \mu \rangle = \|f\|_\infty \|\mu\|_{\mathcal{M}}$.
Since μ is a sum of (scaled) δ-functions, we can choose $f(\boldsymbol{v}) = \pm 1$ at each of these points, choosing the sign of $f(\boldsymbol{v})$ to match the sign of the scaling of the associated δ-function.

We can put $f(x_1, x_2) = x_1 x_2$

Note that this f is Lipschitz with Lipschitz constant $\sqrt{2}$.

In general: if $A = \operatorname{supp} \mu_+$ and $B = \operatorname{supp} \mu_-$ we have a Lipschitz function f where $f(\boldsymbol{x}) = +1$ for $\boldsymbol{x} \in \operatorname{supp} \mu_+$ and $f(\boldsymbol{x}) = -1$ for $\boldsymbol{x} \in \operatorname{supp} \mu_-$:

$$f(\boldsymbol{x}) = \frac{d(\boldsymbol{x}, B) - d(\boldsymbol{x}, A)}{d(\boldsymbol{x}, B) + d(\boldsymbol{x}, A)}$$

$$\operatorname{Lip} f = \frac{2}{\min_{\boldsymbol{a} \in A, \boldsymbol{b} \in B} \|\boldsymbol{a} - \boldsymbol{b}\|}$$

Non-trivial lower bounds

What if there are many vectors in $\mathcal{W}$? How many do we need to get a reasonable approximation?

Choose

$$\boldsymbol{z}_{k+1} \perp \left\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_k, \boldsymbol{w}_{s(k)}, \dots, \boldsymbol{w}_{s(k+1)-1}\right\}$$

where $s(k+1) = s(k) + m - 1 - k$ for $k = 1, 2, \dots, m-2$. Put

$$\widehat{\mu}(\boldsymbol{\xi}) = c \prod_{k=1}^{m-1} \left(\exp(-i\boldsymbol{z}_k^T \boldsymbol{\xi}) - \exp(+i\boldsymbol{z}_k^T \boldsymbol{\xi})\right) \qquad \text{so}$$

$$\mu = c \sum_{\boldsymbol{u} \in \{\pm 1\}^{m-1}} \left(\prod_{k=1}^{m-1} u_k\right) \delta_{\sum_{k=1}^{m-1} u_k \boldsymbol{z}_k}.$$

Choose $\|\boldsymbol{z}_k\|_2 = 1/\sqrt{m-1}$ so that $\left\|\sum_k u_k \boldsymbol{z}_k\right\|_\infty \leq 1$ for all $\boldsymbol{u} \in \{\pm 1\}^{m-1}$.

$$\operatorname{supp} \mu_+ = \left\{ \sum_k u_k \boldsymbol{z}_k \mid \boldsymbol{u} \in \{\pm 1\}^{m-1} \ \& \ \# \{k \mid u_k > 0\} \text{ is even} \right\}$$

$$\operatorname{supp} \mu_- = \left\{ \sum_k u_k \boldsymbol{z}_k \mid \boldsymbol{u} \in \{\pm 1\}^{m-1} \ \& \ \# \{k \mid u_k > 0\} \text{ is odd} \right\}$$

Thus given $\mathcal{W}$ with $|\mathcal{W}| \leq \frac{1}{2} m(m-1)$ there is a function f of Lipschitz constant $\sqrt{m-1}$ with $\|f\|_\infty = 1$ that is unapproximable by span $\left\{\boldsymbol{x} \mapsto \varphi(\boldsymbol{w}^T \boldsymbol{x}) \mid \varphi \in C(\mathbb{R}),\ \boldsymbol{w} \in \mathcal{W}\right\}$.